# 6th lecture

# The inverse of a 3×3 matrix

Suppose we wish to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix}$$

We first place $A$ and $I$ adjacent to each other.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Phase 1

We now proceed by changing the columns of $A$ *left to right* to reduce $A$ to the form $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$

where $*$ can be any number. This form is called **upper triangular**.

First we subtract row 1 from row 2 and twice row 1 from row 3. 'Row' refers to both matrices.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 2 & 7 & 7 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \text{R}_1 \text{ as a reference} \\ R2 - R1 \\ R3 - 2R1 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Now we subtract row 2 from row 3

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{matrix} \text{R}_2 \text{ as a reference} \\ \rightarrow \\ R3 - R2 \end{matrix} \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

## Phase 2

This consists of continuing the row operations to reduce the elements above the leading diagonal to zero.

We proceed *right to left*. We subtract 3 times row 3 from row 1 (the elements in row 2 column 3 is already zero.)

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad R1 - 3R3 \Rightarrow \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$R_3$ as a reference

Finally we subtract 3 times row 2 from row 1.

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad R1 - 3R2 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$R_2$ as a reference

Then we have $A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

(This can be verified by showing that $AA^{-1} = I$ or $A^{-1}A = I$.)

## Matrix Inverse – the Determinant Method

Given a square matrix $A$:

- Find $|A|$. If $|A| = 0$ then $A^{-1}$ does not exist. If $|A| \neq 0$ we can proceed to find the inverse matrix, as follows.
- Replace each element of $A$ by its cofactor (see Section 7.3).
- Transpose the result to form the **adjoint matrix**, denoted by adj$(A)$
- Then calculate $A^{-1} = \dfrac{1}{|A|}\,\text{adj}(A)$.

Find the inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$. This will require five stages.

$$|A| = 0 \times 5 + 1 \times (-1) + 1 \times 7 = 6$$

$$\begin{bmatrix} \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 5 & 1 & 7 \\ -1 & 1 & 1 \\ -4 & -2 & -2 \end{bmatrix}$$

(c) Now attach the signs from the array

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array}$$

Transposing, $\text{adj}(A) = \begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$

Each row become a column

**Answer**

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \frac{1}{6}\begin{bmatrix} 5 & 1 & -4 \\ -1 & 1 & 2 \\ 7 & -1 & -2 \end{bmatrix}$$

# FINITE DIFFERENCES OPERATORS

For a function y=f(x), it is given that $y_0, y_1, ..., y_n$ are the values of the variable y corresponding to the equidistant arguments, $x_0, x_1, ..., x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h,\ x_3 = x_0 + 3h, ..., x_n = x_0 + nh$. In this case, even though Lagrange and divided difference interpolation polynomials can be used for interpolation, some simpler interpolation formulas can be derived. For this, we have to be familiar with some finite difference operators and finite differences, which were introduced by Sir Isaac Newton. Finite differences deal with the changes that take place in the value of a function f(x) due to finite changes in x. Finite difference operators include, forward difference operator, backward difference operator, shift operator, central difference operator and mean operator.

- **Forward difference operator ( $\Delta$ ) :**

For the values $y_0, y_1, ..., y_n$ of a function y=f(x), for the equidistant values $x_0, x_1, x_2, ..., x_n$, where $x_1 = x_0 + h, x_2 = x_0 + 2h,\ x_3 = x_0 + 3h, ..., x_n = x_0 + nh$, the forward difference operator $\Delta$ is defined on the function f(x) as,

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f(x_{i+1}) - f(x_i)$$

That is,

Then, in particular

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0)$$

$$\Rightarrow \quad \Delta y_0 = y_1 - y_0$$

$$\Delta f(x_1) = f(x_1 + h) - f(x_1) = f(x_2) - f(x_1)$$

$$\Rightarrow \quad \Delta y_1 = y_2 - y_1$$

etc.,

$\Delta y_0, \Delta y_1, ..., \Delta y_i, ...$ are known as the **first forward differences.**

The second forward differences are defined as,

$$
\begin{aligned}
\Delta^2 f(x_i) &= \Delta[\Delta f(x_i)] = \Delta[f(x_i+h) - f(x_i)] \\
&= \Delta f(x_i+h) - \Delta f(x_i) \\
&= f(x_i+2h) - f(x_i+h) - [f(x_i+h) - f(x_i)] \\
&= f(x_i+2h) - 2f(x_i+h) + f(x_i) \\
&= y_{i+2} - 2y_{i+1} + y_i
\end{aligned}
$$

In particular,

$$\Delta^2 f(x_0) = y_2 - 2y_1 + y_0 \quad or \quad \Delta^2 y_0 = y_2 - 2y_1 + y_0$$

The third forward differences are,

$$
\begin{aligned}
\Delta^3 f(x_i) &= \Delta[\Delta^2 f(x_i)] \\
&= \Delta[f(x_i+2h) - 2f(x_i+h) + f(x_i)] \\
&= y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i
\end{aligned}
$$

In particular,

$$\Delta^3 f(x_0) = y_3 - 3y_2 + 3y_1 - y_0 \quad or \quad \Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

In general the n[th] forward difference,

$$\Delta^n f(x_i) = \Delta^{n-1} f(x_i + h) - \Delta^{n-1} f(x_i)$$

The differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0 \ldots$ are called the **leading differences.**

]

| x | y | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| $x_0$ | $y_0 = f(x_o)$ | | | |
| | | $\Delta y_0 = y_1 - y_0$ | | |
| $x_1$ | $y_1 = f(x_1)$ | | $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$ | |
| | | $\Delta y_1 = y_2 - y_1$ | | $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$ |
| $x_2$ | $y_2 = f(x_2)$ | | $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ | |
| | | $\Delta y_2 = y_3 - y_2$ | | |
| $x_3$ | $y_3 = f(x_3)$ | | | |

***Example*** Construct the forward difference table for the following $x$ values and its corresponding $f$ values.

| $x$ | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 | 1.1 | 1.3 |
|---|---|---|---|---|---|---|---|
| $f$ | 0.003 | 0.067 | 0.148 | 0.248 | 0.370 | 0.518 | 0.697 |

| $x$ | $f$ | $\Delta f$ | $\Delta^2 f$ | $\Delta^3 f$ | $\Delta^4 f$ | $\Delta^5 f$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.003 | | | | | |
| | | 0.064 | | | | |
| 0.3 | 0.067 | | 0.017 | | | |
| | | 0.081 | | 0.002 | | |
| 0.5 | 0.148 | | 0.019 | | 0.001 | |
| | | 0.100 | | 0.003 | | 0.000 |
| 0.7 | 0.248 | | 0.022 | | 0.001 | |
| | | 0.122 | | 0.004 | | 0.000 |
| 0.9 | 0.370 | | 0.026 | | 0.001 | |
| | | 0.148 | | 0.005 | | |
| 1.1 | 0.518 | | 0.031 | | | |
| | | 0.179 | | | | |
| 1.3 | 0.697 | | | | | |

***Example*** Construct the forward difference table, where $f(x)=\frac{1}{x}$, $x = 1(0.2)2$, 4D.

| $x$ | $f(x)=\frac{1}{x}$ | $\Delta f$ first difference | $\Delta^2 f$ second difference | $\Delta^3 f$ | $\Delta^4 f$ | $\Delta^5 f$ |
|---|---|---|---|---|---|---|
| 1.0 | 1.000 | | | | | |
| | | -0.1667 | | | | |
| 1.2 | 0.8333 | | 0.0477 | | | |
| | | -0.1190 | | -0.0180 | | |
| 1.4 | 0.7143 | | 0.0297 | | 0.0082 | -0.0045 |
| | | -0.0893 | | -0.0098 | | |
| 1.6 | 0.6250 | | 0.0199 | | 0.0037 | |
| | | -0.0694 | | -0.0061 | | |
| 1.8 | 0.5556 | | 0.0138 | | | |
| | | -0.0556 | | | | |
| 2.0 | 0.5000 | | | | | |

***Example*** Construct the forward difference table for the data

$$x: -2 \quad 0 \quad 2 \quad 4$$
$$y = f(x): \quad 4 \quad 9 \quad 17 \quad 22$$

The forward difference table is as follows:

| x | y=f(x) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|
| -2 | 4 | | | |
| | | $\Delta y_0$ =5 | | |
| 0 | 9 | | $\Delta^2 y_0$ =3 | |
| | | $\Delta y_1$ =8 | | $\Delta^3 y_0$ =-6 |
| 2 | 17 | | $\Delta^2 y_1$ =-3 | |
| | | $\Delta y_2$ =5 | | |
| 4 | 22 | | | |