Numerical Analysis

2^{nd} Lecture

2
Bisection And Regular Methods

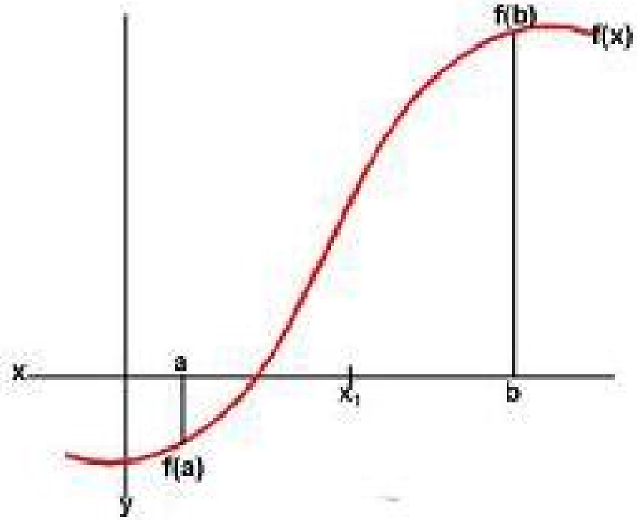
Bisection Method

The bisection method is one of the bracketing methods for finding roots of an equation. For a given a function *f(x)*, guess an interval which might contain a root and perform a number of iterations, where, in each iteration the interval containing the root is get halved.

The **bisection method** is based on the intermediate value theorem for continuous functions.

Intermediate value theorem for continuous functions: If f is a continuous function and $f(a)$ and $f(b)$ have opposite signs, then at least one root lies in between a and b. If the interval (a, b) is small enough, it is likely to contain a single root.

i.e., an interval $[a, b]$ must contain a zero of a continuous function f if the product $f(a)f(b) < 0$. Geometrically, this means that if $f(a)f(b) < 0$, then the curve f has to cross the x-axis at some point in between a and b.



Algorithm : Bisection Method

Suppose we want to find the solution to the equation $f(x) = 0$, where f is continuous.

Given a function $f(x)$ continuous on an interval $[a_0, b_0]$ and satisfying $f(a_0)f(b_0) < 0$.

Example Solve $x^3 - 9x+1 = 0$ for the root between $x = 2$ and $x = 4$, by bisection method.

Given $f(x) = x^3 - 9x + 1$. Now $f(2) = -9$, $f(4) = 29$ so that $f(2)f(4) < 0$ and hence a root lies between 2 and 4.

Set $a_0 = 2$ and $b_0 = 4$. Then

$$x_0 = \frac{(a_0 + b_0)}{2} = \frac{2+4}{2} = 3 \quad \text{and} \quad f(x_0) = f(3) = 1.$$

Since $f(2)f(3) < 0$, a root lies between 2 and 3, hence we set $a_1 = a_0 = 2$ and $b_1 = x_0 = 3$. Then

$$x_1 = \frac{(a_1 + b_1)}{2} = \frac{2+3}{2} = 2.5 \quad \text{and} \quad f(x_1) = f(2.5) = -5.875$$

Since $f(2)f(2.5) > 0$, a root lies between 2.5 and 3, hence we set $a_2 = x_1 = 2.5$ and $b_2 = b_1 = 3$.

Then $x_2 = \frac{(a_2 + b_2)}{2} = \frac{2.5+3}{2} = 2.75$ and $f(x_2) = f(2.75) = -2.9531$.

The steps are illustrated in the following table.

n	x_n	$f(x_n)$
0	3	1.0000
1	2.5	−5.875
2	2.75	−2.9531
3	2.875	−1.1113
4	2.9375	−0.0901

Example Find a real root of the equation $f(x) = x^3 - x - 1 = 0$.

Since $f(1)$ is negative and $f(2)$ positive, a root lies between 1 and 2 and therefore we take $x_0 = 3/2 = 1.5$. Then

$f(x_0) = \frac{27}{8} - \frac{3}{2} = \frac{15}{8}$ is positive and hence $f(1)\, f(1.5) < 0$ and Hence the root lies between 1 and 1.5 and we obtain

$$x_1 = \frac{1 + 1.5}{2} = 1.25$$

$f(x_1) = -19/64$, which is negative and hence $f(1)\, f(1.25) > 0$ and hence a root lies between 1.25 and 1.5. Also,

$$x_2 = \frac{1.25 + 1.5}{2} = 1.375$$

The procedure is repeated and the successive approximations are

$$x_3 = 1.3125, \quad x_4 = 1.34375, \quad x_5 = 1.328125, \text{ etc.}$$

Example Find a positive root of the equation $xe^x = 1$, which lies between 0 and 1.

Let $f(x) = xe^x - 1$. Since $f(0) = -1$ and $f(1) = 1.718$, it follows that a root lies between 0 and 1. Thus,

$$x_0 = \frac{0+1}{2} = 0.5\,.$$

Since $f(0.5)$ is negative, it follows that a root lies between 0.5 and 1. Hence the new root is 0.75, i.e.,

$$x_1 = \frac{.5+1}{2} = 0.75.$$

Since $f(x_1)$ is positive, a root lies between 0.5 and 0.75 . Hence

$$x_2 = \frac{.5+.75}{2} = 0.625$$

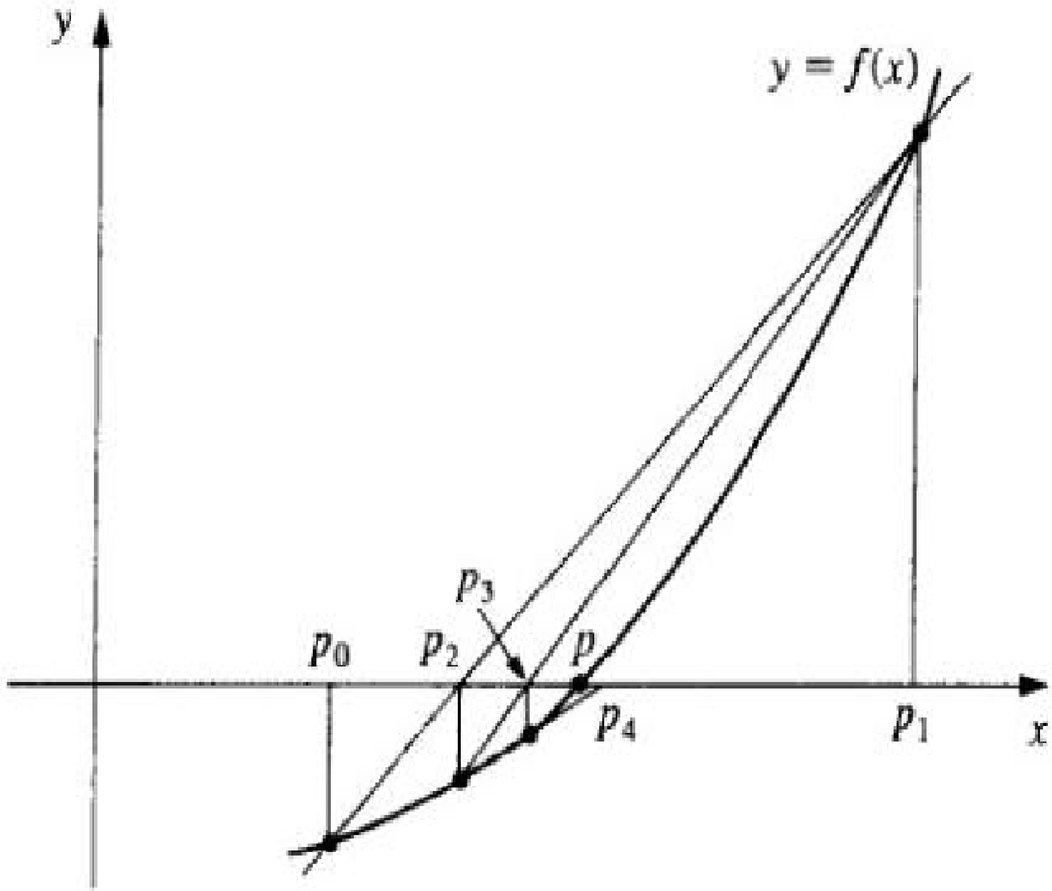
Since $f(x_2)$ is positive, a root lies between 0.5 and 0.625. Hence

$$x_3 = \frac{.5+.625}{2} = 0.5625.$$

We accept 0.5625 as an approximate root.

Regula Falsi method or Method of False Position

This method is also based on the intermediate value theorem. In this method also, as in bisection method, we choose two points a_n and b_n such that $f(a_n)$ and $f(b_n)$ are of opposite signs (i.e., $f(a_n)f(b_n) < 0$). Then, intermediate value theorem suggests that a zero of f lies in between a_n and b_n, if f is a continuous function.



Algorithm: Given a function $f(x)$ continuous on an interval $[a_0\ ,\ b_0]$ and satisfying $f(a_0)f(b_0) < 0$.

For $n = 0, 1, 2, \ldots$ until termination do:

Compute

$$x_n = \frac{\begin{vmatrix} a_n & b_n \\ f(a_n) & f(b_n) \end{vmatrix}}{f(b_n) - f(a_n)}.$$

If $f(x_n) = 0$, accept x_n as a solution and stop.

Else continue.

If $f(a_n)f(x_n) < 0$, set $a_{n+1} = a_n, b_{n+1} = x_n$. Else set $a_{n+1} = x_n, b_{n+1} = b_n$.

Then $f(x) = 0$ for some x in $[a_{n+1}, b_{n+1}]$.

Example Using regula-falsi method, find a real root of the equation,

$f(x) = x^3 + x - 1 = 0$, near $x = 1$.

Here note that $f(0) = -1$ and $f(0) = -1$. Hence $f(0)f(1) < 0$, so by intermediate value theorem a root lies in between 0 and 1. We search for that root by regula falsi method and we will get an approximate root.

Set $a_0 = 0$ and $b_0 = 1$. Then

$$x_0 = \frac{\begin{vmatrix} a_0 & b_0 \\ f(a_0) & f(b_0) \end{vmatrix}}{f(b_0) - f(a_0)} = \frac{\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix}}{1-(-1)} = 0.5$$

$$\frac{((1*0)-(1*-1))}{1+1} = \frac{1}{2}$$

and $f(x_0) = f(0.5) = -0.375$.

Since $f(0)f(0.5) > 0$, a root lies between 0.5 and 1. Set $a_1 = x_0 = 0.5$ and $b_1 = b_0 = 1$.

Then

$$x_1 = \frac{\begin{vmatrix} a_1 & b_1 \\ f(a_1) & f(b_1) \end{vmatrix}}{f(b_1) - f(a_1)} = \frac{\begin{vmatrix} 0.5 & 1 \\ -0.375 & 1 \end{vmatrix}}{1-(-0.375)} = 0.6364$$

$$\frac{((1*0.5)-(-0.375*1))}{1+0.375}$$

and $f(x_1) = f(0.6364) = -0.1058.$

Since $f(0.6364)f(x_1) > 0$, a root lies between x_1 and 1 and hence we set $a_2 = x_1 = 0.6364$ and $b_2 = b_1 = 1.$ Then

$$x_2 = \frac{\begin{vmatrix} a_2 & b_2 \\ f(a_2) & f(b_2) \end{vmatrix}}{f(b_2) - f(a_2)} = \frac{\begin{vmatrix} 0.6364 & 1 \\ -0.1058 & 1 \end{vmatrix}}{1-(-0.1058)} = 0.6712$$

and $f(x_2) = f(0.6712) = -0.0264$

Since $f(0.6712)f(0.6364) > 0,$ a root lies between x_2 and 1, and hence we set $a_3 = x_2 = 0.6364$ and $b_3 = b_1 = 1.$

Then $$x_3 = \frac{\begin{vmatrix} a_3 & b_3 \\ f(a_3) & f(b_3) \end{vmatrix}}{f(b_3) - f(a_3)} = \frac{\begin{vmatrix} 0.6712 & 1 \\ -0.0264 & 1 \end{vmatrix}}{1-(-0.0264)} = 0.6796$$

and $f(x_3) = f(0.6796) = -0.0063 \approx 0$.

Since $f(0.6796) \approx 0.0000$ we accept 0.6796 as an (approximate) solution of $x^3 - x - 1 = 0$.

Example Given that the equation $x^{2.2} = 69$ has a root between 5 and 8. Use the method of regula-falsi to determine it.

Let $f(x) = x^{2.2} - 69$. We find

$$f(5) = -34.50675846 \text{ and } f(8) = +28.00586026.$$

$$x_1 = \frac{\begin{vmatrix} 5 & 8 \\ f(5) & f(8) \end{vmatrix}}{f(8) - f(5)} = \frac{5(28.00586026) - 8(-34.50675846)}{28.00586026 + 34.50675846)} = 6.655990062.$$

Now, $f(x_1) = -4.275625415$ and therefore, $f(5)\, f(x_1) > 0$ and hence the root lies between 6.655990062 and 8.0. Proceeding similarly,

$$x_2 = 6.83400179, \quad x_3 = 6.850669653,$$

The correct root is $x_3 = 6.8523651\cdots$, so that x_3 is correct to these significant figures. We accept 6.850669653 as an approximate root.