3rd lecture

Newton Raphson Method
&
Secant Method

$$x_4 = \frac{x_2 f_3 - x_3 f_2}{f_3 - f_2} = \frac{0.58378(0.00037) - 0.56738(0.02599)}{0.00037 - 0.02599} = \frac{-0.01453}{-0.02562} = 0.5671$$

Approximating to three digits, the root can be considered as 0.567.

The Newton-Raphson method, or Newton Method, is a powerful technique for solving equations numerically. Like so much of the differential calculus, it is based on the simple idea of linear approximation.

Newton – Raphson Method

Consider $f(x)=0$, where f has continuous derivative f'. From the figure we can say that at $x=a,\ y=f(a)=0$; which means that a is a solution to the equation $f(x)=0$. In order to find the value of a, we start with any arbitrary point x_0. From figure we can see that, the tangent to the curve f at $(x_0, f(x_0))$ (with slope $f'(x_0)$) touches the x-axis at x_1.

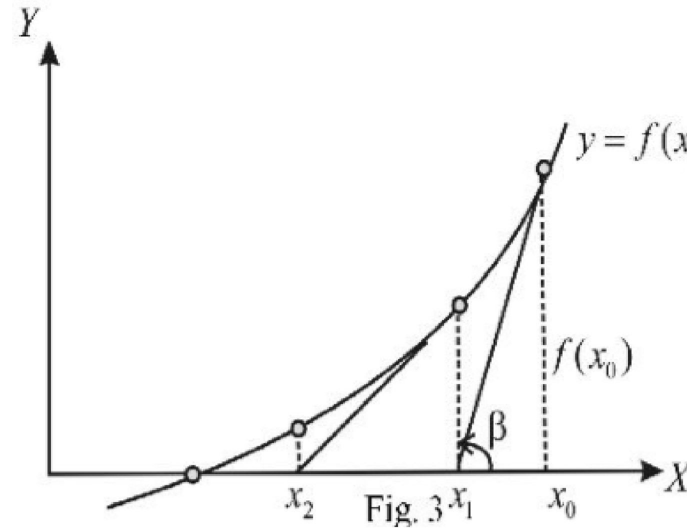


Fig. 3

$$\text{Now, } \tan b = f'(x_0) = \frac{f(x_0) - f(x_1)}{x_0 - x_1},$$

As $f(x_1)=0$, the above simplifies to

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

In the second step, we compute

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)},$$

in the third step we compute

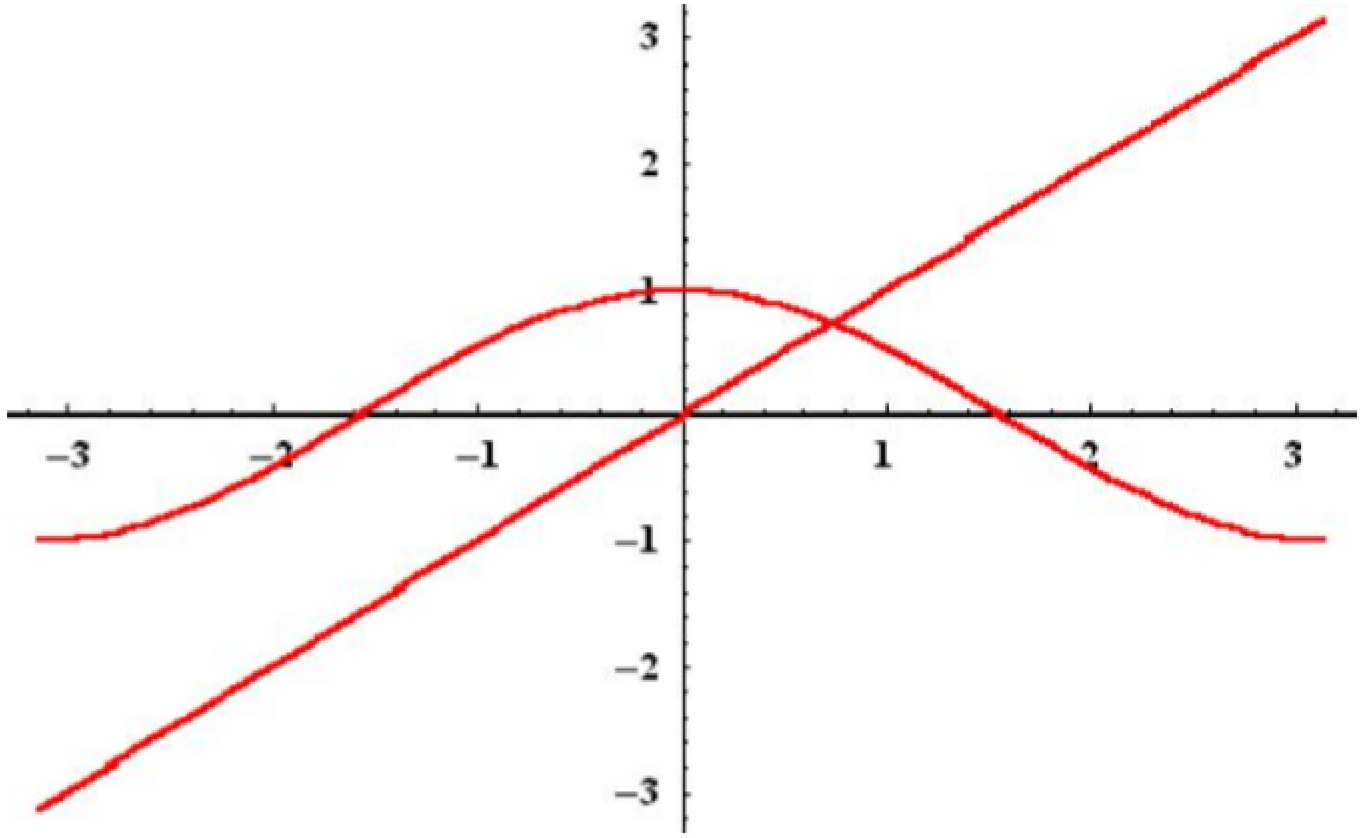
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

and so on. More generally, we write x_{n+1} in terms of x_n, $f(x_n)$ and $f'(x_n)$ for $n = 1, 2, \ldots$ by means of the **Newton-Raphson** formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example. Let us approximate the only solution to the equation $x = \cos x$

In fact, looking at the graphs we can see that this equation has one solution.



This solution is also the only zero of the function $f(x) = x - \cos x$. So now we see how Newton's method may be used to approximate r. Since r is between 0 and $p/2$, we set $x_1 = 1$. The rest of the sequence is generated through the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)} \,.$$

We have

$$\begin{aligned}
x_1 &= 1. \\
x_2 &= 0.750363867840243893034942306682177 \\
x_3 &= 0.739112890911361670360585290904890 \\
x_4 &= 0.739085133385283969760125120856804 \\
x_5 &= 0.739085133215160641661702625685026 \\
x_6 &= 0.739085133215160641655312087673873 \\
x_7 &= 0.739085133215160641655312087673873 \\
x_8 &= 0.739085133215160641655312087673873
\end{aligned}$$

Example Apply Newton's method to solve the algebraic equation $f(x)=x^3+x-1=0$ correct to 6 decimal places. (Start with x_0=1)

$$f(x)=x^3+x-1,$$

$$f'(x)=3x^2+1$$

and substituting these in Newton's iterative formula, we have

$$x_{n+1}=x_n-\frac{x_n^3+x_n-1}{3x_n^2+1} \quad \text{or} \quad x_{n+1}=\frac{2x_n^3+1}{3x_n^2+1},\ n=0,1,2,\ldots.$$

Starting from x_0=1.000 000,

$x_1+0.750000,\ x_2=0.686047,\quad x_3=0.682340,\ x_4=0.682328,\ \cdots$ and we accept 0.682328 as an approximate solution of $f(x)=x^3+x-1=0$ correct to 6 decimal places.

Example Find the positive solution of the transcendental equation

$$2\sin x = x.$$

Here $f(x) = x - 2\sin x$,

so that $f'(x) = 1 - 2\cos x$

Substituting in Newton's iterative formula, we have

$$x_{n+1} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n}, \quad n = 0, 1, 2, \cdots \quad \text{or}$$

$$x_{n+1} = \frac{2(\sin x_n - x_n \cos x_n)}{1 - 2\cos x_n} = \frac{N_n}{D_n}, \quad n = 0, 1, 2, \cdots$$

where we take $N_n = 2(\sin x_n - x_n \cos x_n)$ and $D_n = 1 - 2\cos x_n$, to easy our calculation. Values calculated at each step are indicated in the following table (Starting with $x_0 = 2$).

n	x_n	N_n	D_n	x_{n+1}
0	2.000	3.483	1.832	1.901
1	1.901	3.125	1.648	1.896
2	1.896	3.107	1.639	1.896

1.896 is an approximate solution to $2\sin x = x$.

Example Find a double root of the equation

$$f(x) = x^3 - x^2 - x + 1 = 0.$$

Here $f'(x) = 3x^2 - 2x - 1$, and $f''(x) = 6x - 2$. With $x_0 = 0.8$, we obtain

$$x_0 - 2\frac{f(x_0)}{f'(x_0)} = 0.8 - 2\frac{0.072}{-(0.68)} = 1.012,$$

and

$$x_0 - \frac{f'(x_0)}{f''(x_0)} = 0.8 - \frac{-(0.68)}{2.8} = 1.043,$$

The closeness of these values indicates that there is a doublel root near to unity. For the next approximation, we choose $x_1 = 1.01$ and obtain

$$x_1 - 2\frac{f(x_1)}{f'(x_1)} = 1.01 - 0.0099 = 1.0001,$$

and $$x_1 - \frac{f'(x_1)}{f''(x_1)} = 1.01 - 0.0099 = 1.0001,$$

Hence we conclude that there is a double root at $x = 1.0001$ which is sufficiently close to the actual root unity.

On the other hand, if we apply Newton-Raphson method with $x_0 = 0.8$, we obtain $x_1 = 0.8 + 0.106 \approx 0.91$, and $x_2 = 0.91 + 0.046 \approx 0.96$.

The Secant Method

We have seen that the Newton-Raphson method requires the evaluation of derivatives of the function and this is not always possible, particularly in the case of functions arising in practical problems. In the secant method, the derivative at x_n is approximated by the formula

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}},$$

which can be written as

$$f_n' = \frac{f_n - f_{n-1}}{x_n - x_{n-1}},$$

where $f_n = f(x_n)$. Hence, the Newton-Raphson formula becomes

$$x_{n+1} = x_n - \frac{f_n(x_n - x_{n-1}}{f_n = f_{n-1}} = \frac{x_{n+1} f_n - x_n f_{n-1}}{f_n = f_{n-1}}.$$

It should be noted that this formula requires two initial approximations to the root.

Example Find a real root of the equation $x^3 - 2x - 5 = 0$ using secant method.

Let the two initial approximations be given by $x_{-1} = 2$ and $x_0 = 3$.

We have

$$f(x_{-1}) = f_1 = 8 - 9 = -1, \quad \text{and} \quad f(x_0) = f_0 = 27 - 11 = 16.$$

$$x_1 = \frac{2(16) - 3(-1)}{17} = \frac{35}{17} = 2.058823529.$$

Also,

$$f(x_1) = f_1 = -0.390799923.$$

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{3(-0.390799923) - 2.058823529(16)}{-16.390799923} = 2.08126366.$$

Again

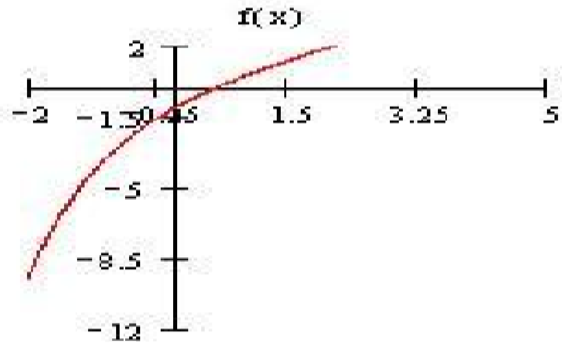
$$f(x_2) = f_2 = -0.147204057.$$

$$x_3 = 2.094824145.$$

Example: Find a real root of the equation $x - e^{-x} = 0$ using secant method.

Solution

The graph of $f(x) = x - e^{-x}$ is as shown here.



Let us assume the initial approximation to the roots as 1 and 2. That is consider $x_{-1} = 1$ and $x_0 = 2$

$$f(x_{-1}) = f_{-1} = 1 - e^{-1} = 1 - 0.367879441 = 0.632120559 \quad \text{and}$$

$$f(x_0) = f_0 = 2 - e^{-2} = 2 - 0.135335283 = 1.864664717.$$

Step 1: Putting $n = 0$, we obtain $x_1 = \dfrac{x_{-1}f_0 - x_0 f_{-1}}{f_0 - f_{-1}}$

Here, $x_1 = \dfrac{1(1.864664717) - 2(0.632120559)}{1.864664717 - 0.632120559} = \dfrac{0.600423599}{1.232544158} = 0.487142.$

Also,

$$f(x_1) = f_1 = 0.487142 - e^{-0.487142} = -0.12724.$$

Step 2: Putting $n = 1$, we obtain

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{2(-0.12724) - 0.487142(1.864664717)}{-0.12724 - 1.864664717} = \frac{-1.16284}{-1.99190} = 0.58378$$

Again

$$f(x_2) = f_2 = 0.58378 - e^{-0.58378} = 0.02599.$$

Step 3: Setting $n = 2$,

$$x_3 = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1} = \frac{0.487142(0.02599) - 0.58378(-0.12724)}{0.02599 - (-0.12724)} = \frac{0.08694}{0.15323} = 0.56738$$

$$f(x_3) = f_3 = 0.56738 - e^{-0.56738} = 0.00037.$$

Step 4: Setting $n = 3$ in (*),

$$x_4 = \frac{x_2 f_3 - x_3 f_2}{f_3 - f_2} = \frac{0.58378(0.00037) - 0.56738(0.02599)}{0.00037 - 0.02599} = \frac{-0.01453}{-0.02562} = 0.5671$$

Approximating to three digits, the root can be considered as 0.567.