5th lecture

Solution of Equations

1 Solution by Cramer's Rule

2 Solution by Gauss Elimination

3 Solution by Inverse Matrix Method

1. Cramer's Rule - two equations

If we are given a pair of simultaneous equations

$$\begin{aligned} a_1 x + b_1 y &= d_1 \\ a_2 x + b_2 y &= d_2 \end{aligned}$$

then x, and y can be found from

$$x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Example

Solve the equations

$$\begin{aligned} 3x + 4y &= -14 \\ -2x - 3y &= 11 \end{aligned}$$

Solution

Using Cramer's rule we can write the solution as the ratio of two determinants.

$$x = \frac{\begin{vmatrix} -14 & 4 \\ 11 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{-2}{-1} = 2, \qquad y = \frac{\begin{vmatrix} 3 & -14 \\ -2 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{5}{-1} = -5$$

The solution of the simultaneous equations is then $x = 2$, $y = -5$.

2. Solving three equations in three unknowns

Cramer's Rule

The solution of the system $\begin{Bmatrix} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{Bmatrix}$ is given by $x=\frac{D_x}{D}$ $y=\frac{D_y}{D}$ $z=\frac{D_z}{D}$

where $D=\begin{vmatrix} a & b & c \\ e & f & g \\ i & j & k \end{vmatrix}$ $D_x=\begin{vmatrix} d & b & c \\ h & f & g \\ l & j & k \end{vmatrix}$ $D_y=\begin{vmatrix} a & d & c \\ e & h & g \\ i & l & k \end{vmatrix}$ $D_z=\begin{vmatrix} a & b & d \\ e & f & h \\ i & j & l \end{vmatrix}$

then x, y and z can be found from

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

ex: $-x + 2y + 3z = -7$

$$-4x - 5y + 6z = -13$$

$$7x - 8y - 9z = 39$$

$$D = \begin{vmatrix} -1 & 2 & 3 \\ -4 & -5 & 6 \\ 7 & -8 & -9 \end{vmatrix} = -1\begin{vmatrix} -5 & 6 \\ -8 & -9 \end{vmatrix} - 2\begin{vmatrix} -4 & 6 \\ 7 & -9 \end{vmatrix} + 3\begin{vmatrix} -4 & -5 \\ 7 & -8 \end{vmatrix} = -1(93) - 2(-6) + 3(67) = \mathbf{120}$$

$$D_x = \begin{vmatrix} -7 & 2 & 3 \\ -13 & -5 & 6 \\ 39 & -8 & -9 \end{vmatrix} = -7\begin{vmatrix} -5 & 6 \\ -8 & -9 \end{vmatrix} - 2\begin{vmatrix} -13 & 6 \\ 39 & -9 \end{vmatrix} + 3\begin{vmatrix} -13 & -5 \\ 39 & -8 \end{vmatrix} = -7(93) - 2(-117) + 3(299) = \mathbf{480}$$

$$D_y = \begin{vmatrix} -1 & -7 & 3 \\ -4 & -13 & 6 \\ 7 & 39 & -9 \end{vmatrix} = -1\begin{vmatrix} -13 & 6 \\ 39 & -9 \end{vmatrix} + 7\begin{vmatrix} -4 & 6 \\ 7 & -9 \end{vmatrix} + 3\begin{vmatrix} -4 & -13 \\ 7 & 39 \end{vmatrix} = -1(-117) + 7(-6) + 3(-65) = \mathbf{-120}$$

$$D_z = \begin{vmatrix} -1 & 2 & -7 \\ -4 & -5 & -13 \\ 7 & -8 & 39 \end{vmatrix} = -1\begin{vmatrix} -5 & -13 \\ -8 & 39 \end{vmatrix} - 2\begin{vmatrix} -4 & -13 \\ 7 & 39 \end{vmatrix} - 7\begin{vmatrix} -4 & -5 \\ 7 & -8 \end{vmatrix} = -1(-299) - 2(-65) - 7(67) = \mathbf{-40}$$

$$x = \frac{480}{120} = 4 \qquad y = \frac{-120}{120} = -1 \qquad z = \frac{-40}{120} = -\frac{1}{3}$$

Answer: (4, -1, -1/3)

Gauss Elimination

Example: Solve the following simultaneous equations

$$2x + 3y + z = 13$$
$$x - y - 2z = -1$$
$$3x + y + 4z = 15$$

Solution: rearrange the equations

$$x - y - 2z = -1 \longrightarrow (1)$$
$$2x + 3y + z = 13 \longrightarrow (2)$$
$$3x + y + 4z = 15 \longrightarrow (3)$$

Step one: Make eq. no ① as refference:

$$x - y - 2z = -1 \quad \rightarrow ④$$

$$5y + 5z = 15 \qquad R2 - 2R1 \rightarrow ⑤$$

$$4y + 10z = 18 \qquad R3 - 3R1 \rightarrow ⑥$$

Step Two:

$$x - y - 2z = -1 \quad \rightarrow ⑦$$

$$5y + 5z = 15 \qquad \text{Refference} \rightarrow ⑧$$

$$6z = 6 \qquad R3 - \frac{4}{5}R2 \rightarrow ⑨$$

Step Three: $6z = 6 \rightarrow z = 1$

From eq. ⑧ $\rightarrow y = 2$

From eq. ⑦ $\rightarrow x = 3$

Step Four: by substituting the values of x, y, and Z of $(3, 2, 1)$ in eq ① or ② or ③ or any one

$$x - y - 2Z = -1$$

$$3 - 2 - 2 = -1$$

2. The inverse of a 2×2 matrix

In this subsection we show how the inverse of a 2×2 matrix can be obtained (if it exists).

Task

Form the matrix products AB and BA where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Your solution

$AB =$ $\qquad\qquad$ $BA =$

Answer

$$AB = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (ad - bc)I$$

$$BA = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (ad - bc)I$$

You will see that had we chosen $C = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ instead of B then both products AC and CA will be equal to I. This requires $ad - bc \neq 0$. Hence this matrix C is the inverse of A. However, note, that if $ad - bc = 0$ then A has **no inverse**. (Note that for the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, which occurred in the last task, $ad - bc = 1 \times 0 - 0 \times 2 = 0$ confirming, as we found, that A has no inverse.)

Key Point 8

The Inverse of a 2×2 Matrix

If $ad - bc \neq 0$ then the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a (unique) inverse given by

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note that $ad - bc = |A|$, the determinant of the matrix A.

In words: To find the inverse of a 2×2 matrix A we interchange the diagonal elements, change the sign of the other two elements, and then divide by the determinant of A.

Which of the following matrices has an inverse?

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Your solution

Answer

$|A| = 1 \times 3 - 0 \times 2 = 3; \quad |B| = 1 + 1 = 2; \quad |C| = 2 - 2 = 0; \quad |D| = 1 - 0 = 1.$

Therefore, A, B and D each has an inverse. C does not because it has a zero determinant.

Task

Find the inverses of the matrices A, B and D in the previous Task.

se Key Point 8:

Your solution

$A^{-1} =$ $\qquad$ $B^{-1} =$ $\qquad$ $C^{-1} =$

Answer

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix},\ B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},\ D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = D$$

It can be shown that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ represents an **anti-clockwise** rotation through an angle θ in an xy-plane about the origin. The matrix B represents a rotation **clockwise** through an angle θ. It is given therefore by

$$B = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$