Numerical Analysis

Chemical Engineering & Petroluem Industreis Department

Third Year

NUMERICAL METHODS

VI SEMESTER

CORE COURSE

B Sc MATHEMATICS

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INTRODUCTION

- **Numerical analysis** is the study of algorithm that use numerical approximation for the problems of mathematical analysis.
- The goal of the field of numerical analysis is the design and analysis of techniques to give approximate but accurate solutions to hard problems

Error in Numerical Computation

Due to errors that we have just discussed, it can be seen that our numerical result is an approximate value of the (sometimes unknown) exact result, except for the rare case where the exact answer is sufficiently simple rational number.

If $\tilde{a}$ is an approximate value of a quantity whose exact value is a, then the difference $\varepsilon = \tilde{a} - a$ is called the absolute error of $\tilde{a}$ or, briefly, the error of $\tilde{a}$. Hence, $\tilde{a} = a + \varepsilon$, i.e.

Approximate value = True value + Error.

For example, if $\tilde{a} = 10.52$ is an approximation to $a = 10.5$, then the error is $\varepsilon = 0.02$. The relative error, ε_r, of $\tilde{a}$ is defined by

$$|\varepsilon_r| = \frac{|\varepsilon|}{|a|} = \frac{|\text{Error}|}{|\text{Truevalue}|}$$

For example, consider the value of $\sqrt{2}(=1.414213\ldots)$ up to four decimal places, then

$$\sqrt{2} = 1.4142 + \text{Error}.$$

$$|\text{Error}| = |1.4142 - 1.41421| = .00001,$$

taking 1.41421 as true or exact value. Hence, the relative error is

$$\varepsilon_r = \frac{0.00001}{1.4142}.$$

In this chapter and in the coming chapters, we present the following indirect or iterative methods with illustrative examples:

1. Fixed Point Iteration Method

2. Bisection Method

3. Method of False Position (Regula Falsi Method)

4. Newton-Raphson Method (Newton's method)

1- Fixed Point Method

Example Solve $f(x) = x^2 - 3x + 1 = 0$, by fixed point iteration method.

Solution

Write the given equation as

$x^2 = 3x - 1$ or $x = 3 - 1/x$.

Choose $f(x) = 3 - \frac{1}{x}$. Then $f'(x) = \frac{1}{x^2}$ and $|f'(x)| < 1$ on the interval $(1, 2)$.

Hence the iteration method can be applied to the Eq. (3).

The iterative formula is given by

$$x_{n+1} = 3 - \frac{1}{x_n} \qquad (n = 0, 1, 2, \ldots)$$

Starting with, $x_0 = 1$, we obtain the sequence

$x_0 = 1.000$, $x_1 = 2.000$, $x_2 = 2.500$, $x_3 = 2.600$, $x_4 = 2.615$, . . .

Example Find a real root of the equation $x^3 + x^2 - 1 = 0$ on the interval $[0, 1]$ with an accuracy of 10^{-4}.

To find this root, we rewrite the given equation in the form

$$x = \frac{1}{\sqrt{x+1}}$$

Take

$$f(x) = \frac{1}{\sqrt{x+1}}. \text{ Then } f'(x) = -\frac{1}{2}\frac{1}{(x+1)^{\frac{3}{2}}}$$

$$\max_{[0,1]} |f'(x)| = \left|\frac{1}{2\sqrt{8}}\right| = k = 0.17678 < 0.2.$$

Choose $f(x) = 3 - \frac{1}{x}$. Then $f'(x) = \frac{1}{x^2}$ and $|f'(x) <| 1$ on the interval $(1, 2)$.

Hence the iteration method gives:

n	x_n	$\sqrt{x_n + 1}$	$x_{n+1} = 1/\sqrt{x_n + 1}$
0	0.75	1.3228756	0.7559289
1	0.7559289	1.3251146	0.7546517
2	0.7546617	1.3246326	0.7549263

At this stage,

$$|x_{n+1} - x_n| = 0.7549263 - 0.7546517 = 0.0002746,$$

Example Use the method of iteration to find a positive root, between 0 and 1, of the equation $xe^x = 1$.

Writing the equation in the form

We find that $f(x) = e^{-x}$ and so $f'(x) = -e^{-x}$.

Hence $|f'(x)| < 1$ for $x < 1$, which assures that the iterative process defined by the equation $x_{n+1} = f(x_n)$ will be convergent, when $x < 1$.

The iterative formula is

$$x_{n+1} = \frac{1}{e^{x_n}} \qquad (n = 0, 1, \dots)$$

Starting with $x_0 = 1$, we find that the successive iterates are given by

$$x_1 = 1/e = 0.3678794, \; x_2 = \frac{1}{ex_1} = 0.6922006,$$

$$x_3 = 0.5004735, \qquad x_4 = 0.6062435,$$

$$x_5 = 0.5453957, \qquad x_6 = 0.5796123,$$

We accept 6.5453957 as an approximate root.

Example **Find a solution of $f(x)=x^3+x-1=0$, by fixed point iteration.**

$x^3 + x - 1 = 0$ can be written as $x(x^2+1)=1$, or $x=\dfrac{1}{x^2+1}$.

Note that

$$|f'(x)| = \frac{2|x|}{(1+x^2)^2} < 1 \text{ for any real } x,$$

so by the Theorem we can expect a solution for any real number x_0 as the starting point.

Choosing $x_0 = 1$, and undergoing calculations in the iterative formula

$$x_{n+1} = f(x_n) = \frac{1}{1+x_n^2} \quad (n = 0, 1, \ldots), \qquad \ldots(4)$$

we get the sequence

x_0=1.000 , x_1=0.500, x_2=0.800, x_3 =0.610 ,

x_4= 0.729, x_5=0.653, x_6=0.701, ...

and we choose 0.701 as an (approximate) solution to the given equation.

Example Find the root of the equation $2x = \cos x + 3$ correct to three decimal places.

We rewrite the equation in the form

$$x = \frac{1}{2}(\cos x + 3)$$

so that

$$f = \frac{1}{2}(\cos x + 3),$$

and

$$| f(x) | = \left| \frac{\sin x}{2} \right| < 1.$$

Hence the iteration method can be applied to the eq. (3) and we start with $x_0 = p/2$. The successive iterates are

$$x_1 = 1.5, \quad x_2 = 1.535, \quad x_3 = 1.518,$$
$$x_4 = 1.526, \quad x_5 = 1.522, \quad x_6 = 1.524,$$
$$x_7 = 1.523, \quad x_8 = 1.524.$$

We accept the solution as 1.524 correct to three decimal places.