8th lecture

SOLUTION BY ITERATION:
Jacobi's iteration method
and
Gauss Seidel iteration method

(1) The Jacobi Method

***Two assumptions made on Jacobi Method*:**

1. The system given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n \end{aligned}$$

Has a unique solution.

2. The coefficient matrix A has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

Main idea of Jacobi

To begin, solve the 1st equation for x_1, the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})$$

Then make an initial guess of the solution $\boldsymbol{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots x_n^{(0)})$. Substitute these values into the right hand the rewritten equations to obtain the *first approximation,* $\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots x_n^{(1)}\right)$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots x_n^{(2)}\right)$ is computed by substituting the first approxi vales into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\boldsymbol{x}^{(k)} = \left(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots x_n^{(k)}\right)^t, \quad k = 1,2,3, \dots$

Example. Apply the Jacobi method to solve

$$5x_1 - 2x_2 + 3x_n = -1$$
$$-3x_1 + 9x_2 + x_n = 2$$
$$2x_1 - x_2 - 7x_n = 3$$

Continue iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, rewrite the system

$$x_1 = \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$
$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$
$$x_3 = -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$$

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

The first approximation is

$$x_1^{(1)} = \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200$$

$$x_2^{(1)} = \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222$$

$$x_3^{(1)} = -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429$$

2//

$$x_1 = \frac{-1}{5} + \frac{2}{5}(0.222) - \frac{3}{5}(-0.429)$$

$$= -0.2 + 0.0888 + 0.2574$$

$$= 0.146$$

$$x_2 = \frac{2}{9} + \frac{3}{9}\overset{x_1}{(-0.2)} - \frac{1}{9}\overset{x_3}{(-0.429)}$$

$$= 0.2222 + 0.06666 + 0.0476667$$

$$= 0.203$$

$$x_3 = -\frac{3}{7} + \frac{2}{7}(-0.2) - \frac{1}{7}(0.222)$$

$$= -0.42857 - 0.057143 - 0.031714$$

$$= -0.51743$$

3//

$$x_1 = -\frac{1}{5} + \frac{2}{5}(0.203) - \frac{3}{5}(-0.51743)$$

$$= 0.192$$

$$x_2 = \frac{2}{9} + \frac{3}{9}(0.146) - \frac{1}{9}(-0.51743)$$

$$= 0.3288$$

$$x_3 = -\frac{3}{7} + \frac{2}{7}(0.146) - \frac{1}{7}(-0.51743)$$

Continue iteration, we obtain

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$x_1^{(k)}$	0.000	-0.200	0.146	0.192			
$x_2^{(k)}$	0.000	0.222	0.203	0.328			
$x_2^{(k)}$	0.000	-0.429	-0.517	-0.416			

(2) **Gauss Seidel iteration method**

A simple modification to Jacobi's iteration method is given by *Gauss-Seidel* method.

Step 1 (*Gauss-Seidel method*): Determination of first approximation $x_1^{(1)}, x_2^{(1)}, \dots, x_n^{(1)}$ using $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$.

$$\left.\begin{aligned}
x_1^{(1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(0)} - \frac{a_{13}}{a_{11}} x_3^{(0)} - \cdots - \frac{a_{1n}}{a_{11}} x_n^{(0)} \\
x_2^{(1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \frac{a_{23}}{a_{22}} x_3^{(0)} - \cdots - \frac{a_{2n}}{a_{22}} x_n^{(0)} \\
x_3^{(1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(1)} - \frac{a_{32}}{a_{33}} x_2^{(1)} - \cdots - \frac{a_{2n}}{a_{33}} x_n^{(0)} \\
&\vdots \\
x_n^{(1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(1)} - \frac{a_{n2}}{a_{nn}} x_2^{(1)} - \cdots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(1)}
\end{aligned}\right\} \quad \dots (5)$$

Step $n+1$: In general, if $x_1^{(n)}, x_2^{(n)}, \dots, x_n^{(n)}$ are a system of nth approximations, then the next approximation is given by the formula

$$\left.\begin{aligned}
x_1^{(n+1)} &= \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(n)} - \frac{a_{13}}{a_{11}} x_3^{(n)} - \quad \cdots \quad - \frac{a_{1n}}{a_{11}} x_n^{(n)} \\
x_2^{(n+1)} &= \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(n+1)} - \frac{a_{23}}{a_{22}} x_3^{(n)} - \quad \cdots \quad - \frac{a_{2n}}{a_{22}} x_n^{(n)} \\
x_3^{(n+1)} &= \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(n+1)} - \frac{a_{32}}{a_{33}} x_2^{(n+1)} - \quad \cdots \quad - \frac{a_{2n}}{a_{33}} x_n^{(n)} \\
&\vdots \\
x_n^{(n+1)} &= \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(n+1)} - \frac{a_{n2}}{a_{nn}} x_2^{(n+1)} - \quad \cdots \quad - \frac{a_{n,n-1}}{a_{nn}} x_{n-1}^{(n+1)}
\end{aligned}\right\} \quad \ldots \quad (6)$$

(6) can be briefly described as follows:

$$x_i^{(r+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{(r+1)} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{(r)} \quad (r = 0, 1, 2, \ldots, \quad i = 1, 2, \ldots, n).$$

Example **11** Solve the following system of equations using (a) Jacobi's iteration method and (b) Gauss-Seidel iteration method.

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$

$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$

$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$

$$-x_1 - x_2 - 2x_3 + 10x_4 = -9.$$

Solution

To solve these equations by the iterative methods, we re-write them as follows:

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

It can be verified that these equations satisfy the diagonal dominance condition. The process and given in the following Tables.

Table 1. Jacobi's Method

n	x_1	x_2	x_3	x_4
1	0.3	1.5	2.7	−0.9
2	0.78	1.74	2.7	−0.18
3	0.9	1.908	2.916	−0.108
4	0.9624	1.9608	2.9592	−0.036
5	0.9845	1.9848	2.9851	−0.0158
6	0.9939	1.9938	2.9938	−0.006
7	0.9975	1.9975	2.9976	−0.0025
8	0.9990	1.9990	2.9990	−0.0010
9	0.9996	1.9996	2.9996	−0.0004
10	0.9998	1.9998	2.9998	−0.0002
11	0.9999	1.9999	2.9999	−0.0001
12	1.0	2.0	3.0	0.0

Jacobi method

$$x_1 = 0.3 + 0.2\,x_2 + 0.1\,x_3 + 0.1\,x_4$$

$$x_2 = 1.5 + 0.2\,x_1 + 0.1\,x_3 + 0.1\,x_4$$

$$x_3 = 2.7 + 0.1\,x_1 + 0.1\,x_2 + 0.2\,x_4$$

$$x_4 = -0.9 + 0.1\,x_1 + 0.1\,x_2 + 0.2\,x_3$$

(1)

$x_1 = 0.3$

$x_2 = 1.5$

$x_3 = 2.7$

$x_4 = -0.9$

(2)

$x_1 = 0.3 + 0.2(1.5) + 0.1(2.7) + 0.1(-0.9)$

$= 0.78$

$x_2 = 1.5 + 0.2(0.3) + 0.1(2.7) + 0.1(-0.9)$

$= 1.74$

$x_3 = 2.7 + 0.1(0.3) + 0.1(1.5) + 0.2(-0.9)$

$= 2.7$

$x_4 = -0.9 + 0.1(0.3) + 0.1(1.5) + 0.2(2.7)$

$= -0.18$

(3)

$$x_1 = 0.3 + 0.2(1.74) + 0.1(2.7) + 0.1(-0.18)$$
$$= 0.9$$

$$x_2 = 1.5 + 0.2(0.78) + 0.1(2.7) + 0.1(-0.18)$$
$$= 1.908$$

$$x_3 = 2.7 + 0.1(0.78) + 0.1(1.74) + 0.2(-0.18)$$
$$= 2.916$$

$$x_4 = -0.9 + 0.1(0.78) + 0.1(1.74) + 0.2(2.7)$$
$$= -0.108$$

Table 2. Gauss-Seidel method

n	x_1	x_2	x_3	x_4
1	0.3	1.56	2.886	−0.1368
2	0.8869	1.9523	2.9566	−0.0248
3	0.9836	1.9899	2.9924	−0.0042
4	0.9968	1.9982	2.9987	−0.0008
5	0.9994	1.9997	2.9998	−0.0001
6	0.9999	1.9999	3.0	0.0
7	1.0	2.0	3.0	0.0

From Tables 1 and 2, it is clear that twelve iterations are required by Jacobi's method to achieve the same accuracy as seven Gauss-Seidel iterations.

Gauss-Seidel method :-

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4$$

$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4$$

$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4$$

$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3$$

الحلقة الاولى

$n=1$

$$x_1 = 0.3$$

$$x_2 = 1.5 + 0.2(0.3) + 0 + 0$$
$$= 1.56$$

$$x_3 = 2.7 + 0.1(0.3) + 0.1(1.56) + 0$$
$$= 2.886$$

$$x_4 = -0.9 + 0.1(0.3) + 0.1(1.56) + 0.2(2.886)$$
$$= -0.1368$$

المحاولة الثانية -

$n=2$

$$x_1 = 0.3 + 0.2(1.56) + 0.1(2.886) + 0.1(-0.1368)$$
$$= 0.8869$$

$$x_2 = 1.5 + 0.2(0.8869) + 0.1(2.886) + 0.1(-0.13$$
$$= 1.9523$$

$$x_3 = 2.7 + 0.1(0.8869) + 0.1(1.9523) + 0.2(-0.1368)$$
$$= 2.9566$$

$$x_4 = -0.9 + 0.1(0.8869) + 0.1(1.9523) + 0.2(2.9566)$$
$$= -0.0248$$

المحاولة الثالثة

$n=3$

$$x_1 = 0.3 + 0.2(1.9523) + 0.1(2.9566) + 0.1(-0.0248)$$
$$= 0.9836$$

Home work

1. Solve by Jacobi's iteration method, the system of equations

$$20x_1 + x_2 - 7x_3 = 17$$

$$3x_1 + 20x_2 - x_3 = -18$$

$$2x_1 - 3x_2 + 20x_3 = 25$$

2. Using Gauss Siedel iteration solve the following system of equations

$$10x + 2y + z = 9$$

$$2x + 20y - 2z = -44$$

$$-2x + 3y + 10z = 22$$