Al-Mustaqbal University College

Department of Medical Instrumentation Technologies

Mathematics II / Second Stage

By

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Taylor's Series (Power Series about $x=a$)

Let $f(x)$ be a continuous function with derivatives of all orders are exists at $(x=a)$, then the Taylor series generated by $f(x)$ @ $x=a$ is :

$$\left[\sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x-a)^{n} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^{2} + \cdots + \frac{f^{n}(a)}{n!}(x-a)^{n} + \cdots \quad \text{—③}\right]$$

It's clear that, like in Maclaurin's series, in Taylor's series $f(x)$ can be represented as follows :

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n + \cdots \quad \text{—④}$$

Now to find the coefficients $(a_0, a_1, \cdots, a_n)$, we

follow the same procedure like in Maclaurin's series (from eq.1 to eq.2) but with replacing $(x=0)$ by $(x=a)$, got the Taylor polynomial of order n about $x=a$:

$$\left[f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{n}(a) + \cdots \quad \text{—⑤}\right]$$

$$** \sim\sim\sim **$$

Ex) Find Taylor's series for $f(x) = \ln x$ about $x = 1$?

Sol.) $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$ (eq-5)

if $a = 1 \rightarrow$

$$f(1) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots$$

$$f(x) = \ln x \quad \rightarrow f(1) = \ln 1 = 0$$
$$f'(x) = 1/x \quad \rightarrow f'(1) = 1$$
$$f''(x) = -(1/x^2) \quad \rightarrow f''(1) = -1$$
$$f'''(x) = (2/x^3) \quad \rightarrow f'''(1) = 2$$
$$f^{IV}(x) = (-6/x^4) \quad \rightarrow f^{IV}(1) = -6 \quad \rightarrow$$

$$\ln x = 0 + (x-1) - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \cdots$$
$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$**\sim\sim\sim**$

Ex: Find Taylor series for $f(x) = \sin x$ about $x = \frac{\pi}{6}$?

Sol: $f(x) = f(\frac{\pi}{6}) + (x-\frac{\pi}{6}) f'(\frac{\pi}{6}) + \frac{(x-\pi/6)}{2!} f''(\frac{\pi}{6}) + \cdots$

$f(x) = \sin x \quad \rightarrow \quad f(\pi/6) = \sin \pi/6 = \frac{1}{2}$

$f'(x) = \cos x \quad \rightarrow \quad f'(\pi/6) = \sqrt{3}/2$

$f''(x) = -\sin x \quad \rightarrow \quad f''(\pi/6) = -1/2$

$f'''(x) = -\cos x \quad \rightarrow \quad f'''(\pi/6) = -\sqrt{3}/2$ $\rightarrow$ by Sub. in eq. above

$$\sin x = \frac{1}{2} + (x-\frac{\pi}{6})\frac{\sqrt{3}}{2} + \frac{(x-\frac{\pi}{6})^2}{2!}(-\frac{1}{2}) + \frac{(x-\frac{\pi}{6})^3}{3!}(-\sqrt{3}/2) + \cdots$$

$$= \frac{1}{2} + (x-\frac{\pi}{6})\frac{\sqrt{3}}{2} - \frac{(x-\pi/6)^2}{2*2!} - \frac{\sqrt{3}}{2*3!}(x-\frac{\pi}{6})^3 + \cdots$$

$**\sim\sim\sim**$

Ex) Find Taylor's series for $f(x) = \sqrt{x}$ about $x = 4$?

Sol.) using eq. 5 $\rightarrow$

$$f(x) = f(4) + (x-4)f'(4) + \frac{(x-4)^2}{2!}f''(4) + \frac{(x-4)^3}{3!}f'''(4) + \cdots$$

hence,

$f(x) = \sqrt{x} \rightarrow f(4) = 2$

$f'(x) = 1/2\sqrt{x} \rightarrow f'(4) = 1/4$

$f''(x) = -1/4x^{3/2} \rightarrow f''(4) = -1/32$

$f'''(x) = 3/8x^{5/2} \rightarrow f'''(4) = 3/256 \longrightarrow$

$$f(x) = \sqrt{x} = 2 + (x-4)\left(\frac{1}{4}\right) + \frac{(x-4)^2}{2!}\left(-\frac{1}{32}\right) + \frac{(x-4)^3}{3!}\left(\frac{3}{256}\right) + \cdots$$

$$\therefore \sqrt{x} = 2 + \frac{(x-4)}{4} - \frac{(x-4)^2}{64} + \frac{(x-4)^3}{512} \cdots$$