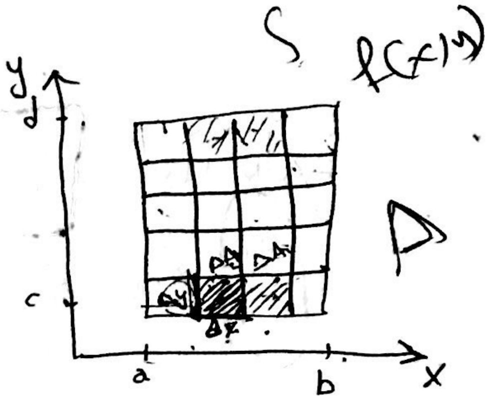
« multiple Integrals »

D. Double Integral : DI

suppose that $f(x,y)$, is defined on a rectangular region R given by $R: a \le x \le b$, $c \le y \le d$

we imagine R to be covered by network of lines parallel to the x & y axis, these lines divide R into small piece of area.

$$\boxed{\Delta A_1 = \Delta x \, \Delta y}$$



we number these in some order

$\Delta A_1 , \Delta A_2 , \dots$

ΔA_n, chose a point (x_k, y_k) in each piece.

ΔA_k, and From the Sum :-

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \, \Delta A_k$$

if f is continous through R, then $(\Delta x, \Delta y)$ go to the zero

So :-

$$S_n = \sum_{k=1}^{n} f(x_k, y_k) \, \Delta A_k = \iint_R f(x,y) \, dA = \iint_R f(x,y) \, dx \, dy$$

« properties of DI »

1) $\iint_R k f(x,y) \, dA = k \iint_R f(x,y) \, dA$ where k is any no.

$$\iint_R [f(x,y) + g(x,y)] = \iint_R f(x,y) \, dA + \iint_R g(x,y) \, dA$$

③ $\iint_R [f(x,y)-g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$

④ $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R

⑤ $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R

⑥ $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$

Ex_1// Evaluate $\int_0^3 \boxed{\int_0^2 (4-y^2)\,dy}\,dx$

solution

$$\int_0^3 \left(4y-\frac{y^3}{3}\right]_0^2 dx = \int_0^3 [8-\frac{8}{3}]-[0]\,dx = \int_0^3 \frac{16}{3}\,dx$$

$y^0 = 1$

$$= \frac{16}{3}x\Big]_0^3 = 16$$

Ex_2// Calculate $\iint_R (1-6x^2y)\,dA$ if $R: 0 \leq x \leq 2$, $-1 \leq y \leq 1$?

$6\frac{x^3}{3}$

Solution

$$\int_{-1}^1 \boxed{\int_0^2 (1-6x^2y)\,dx}\,dy = \int_{-1}^1 [x-2x^3y]_0^2\,dy$$

$$= \int_{-1}^1 [2-16y]-[0]\,dy = \int_{-1}^1 [2-16y]\,dy = 2y-8y^2\Big]_{-1}^1$$

$$= 2-8-[-2-8] = 2-8+2+8 = 4$$

or $\int_0^2\int_{-1}^1 [1-6x^2y]\,dy\,dx = \int_0^2 (y-3x^2y^2\Big]_{-1}^1 dx$

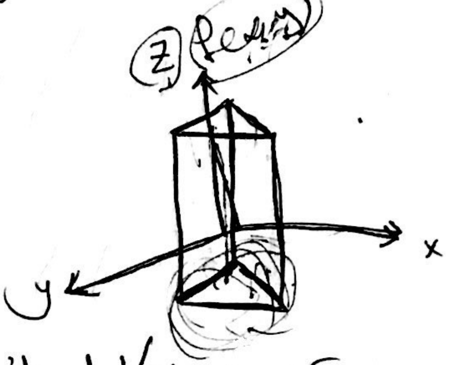
$$= \int_0^2 (1-3x^2)-(-1-3x^2)\,dx = \int_0^2 (1-3x^2+1+3x^2)\,dx$$

$$= 2\int_0^2 dx = 2x\Big]_0^2 = 4$$

Note :- ملاحظات

① $\int_a^b \int_c^d f(x,y)\, \underset{dx\,dy}{dA} = \int_c^d \int_a^b f(x,y)\,dy\,dx$ [Fubini's theorm]

② when $f(x,y)$ is positive, we may interpert the double integral of "f" over region "R" as the volume of the solid prism (موشور) bounded below by "R" & above by (الارتفاع) the surface $z = f(x,y)$



$$\text{Volume} = \iint_R z(x,y)\,dy\,dx$$

* يمكن من التكامل الثنائي ايجاد الحجم بشرط ان x , y تمثل المساحة و $z = f(x,y)$ والارتفاع هو z

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Ex3// Find the volume under plane $z = 4 - x - y$ over the region $R: 0 \le x \le 2,\ 0 \le y \le 1$?

· Solution

$$V = \int_0^2 \int_0^1 \underbrace{(4 - x - y)}_{f(x,y)}\,dy\,dx = \int_0^2 \left(4y - xy - \frac{y^2}{2}\right)\Big]_0^1 dx$$

$$= \int_0^2 \left(4 - x - \frac{1}{2}\right)dx = 4x - \frac{x^2}{2} - \frac{x}{2}\Big]_0^2 = 8 - 2 - 1 = 5 \text{ cubic unit}$$

Ex4// evaluate $\int_\pi^{2\pi} \int_0^\pi (\sin x + \cos y)\,dx\,dy$

Solution

$$\int_\pi^{2\pi} -\cos x + x\cos y\Big]_0^\pi dy = \int_\pi^{2\pi} (1 + \pi\cos y) - (-1 + 0)\,dy$$

$$= \int_\pi^{2\pi} (2 + \pi\cos y)\,dy = 2y + \pi\sin y\Big]_\pi^{2\pi}$$

$$= [4\pi + 0] - [2\pi + 0] = 2\pi$$

* Fubini's theorm:-

let $f(x,y)$ be continous on a region R

① if R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$

then $\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g_1}^{g_2} f(x,y)\,dy\,dx$

② if R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$ then

$$\iint\limits_R f(x,y)\,dA = \int_c^d \int_{h_1}^{h_2} f(x,y)\,dx\,dy$$

Ex5// evaluate $\int_0^{\pi} \int_0^{x} x \sin y \,dy\,dx$

solution

$\int_0^{\pi} -x\cos y \Big]_0^{x} dx = \int_0^{\pi} (-x\cos x + x)\,dx$

$= \int_0^{\pi} -x\cos x\,dx + \int_0^{\pi} x\,dx$

هذا الطرف هو حاصل ضرب
دالتين لذلك يحل لوحده
بطريقة udu لانه لايمكن
حله بالطرق الاعتيادية

let $u = -x$ $\quad dv = \cos x$.

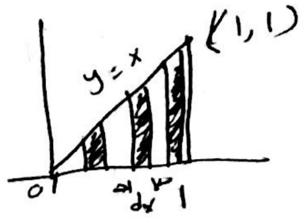
$du = -dx$ $\quad v = +\sin x$

$= uv - \int v\,du$

$= -x\sin x \Big]_0^{\pi} - \int_0^{\pi} -\sin x\,dx + \int_0^{\pi} x\,dx$

$= 0 - [\cos x]_0^{\pi} + \frac{x^2}{2}\Big]_0^{\pi} = 0 + 1 + 1 + \frac{\pi^2}{2} = 2 + \frac{\pi^3}{2}$

Ex6 // Find volume of the prism whose base is the triangle in the x-y plane bounded by the x-axis and the line $y=x$ & $x=1$ and whose top lies in the plane $z=f(x,y)=3-x-y$



Solution

$$V=\iint_R z(x,y)\,dA \Rightarrow V=\int_0^1\int_0^x (3-x-y)\,dy\,dx$$

* قاعدة الموشور هي مثلث ارتفاعه من (0 ← X)

$$V=\int_0^1 3y-xy-\frac{y^2}{2}\Big]_0^x dx=\int_0^1\left(3x-x^2-\frac{x^2}{2}\,dx\right)$$

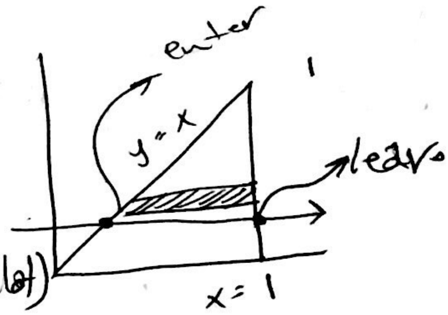
$$V=\int_0^1 3x\,dx-\int_0^1 \frac{3}{2}x^2\,dx \Rightarrow \frac{3}{2}x^2\Big]_0^1-\frac{x^3}{2}\Big]_0^1$$

$$=\frac{3}{2}-\frac{1}{2}=1 \text{ cubic unit.}$$

Ex7 // calculate $\iint_R \frac{\sin x}{x}\,dA$, R is the triangle bounded by x-axis line $y=x$ and the line $x=1$

Solution

$$\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$$

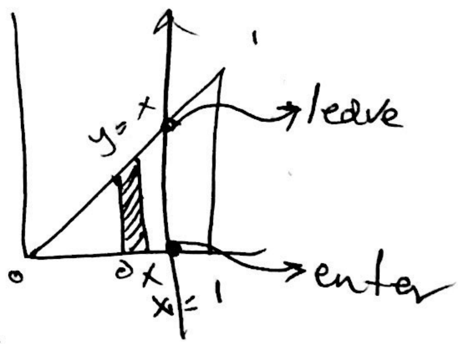


the $\int \frac{\sin x}{x}\,dx$ (cannot be calculated)

So //

$$\int_0^1\int_0^x \frac{\sin x}{x}\,dy\,dx$$

$$\int_0^1 y\,\frac{\sin x}{x}\Big]_0^x dx$$



ملاحظة // لا يمكن تكامل الحالة الاولى لذلك نقلب حدود التكامل لكي نستطيع حلها.

(5)

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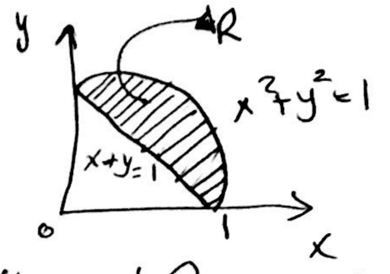
* Finding the limits of integration :-

A. To evaluate $\iint_R f(x,y)\,dA$ over a region R integration First with respect to (y) and then with respect to (x) take the following steps :-

① Sketch the region of integration and label the bounding curves

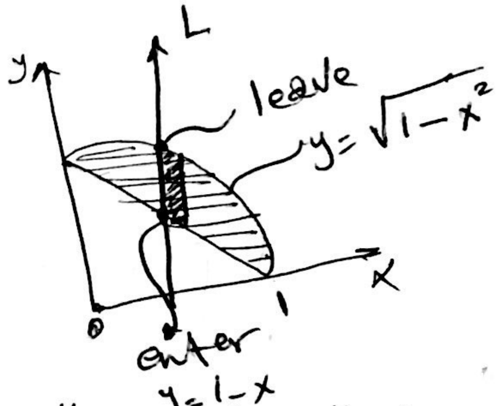


* نرسم الحدود للدالة المعطاة ونحدد .R. الواقعة بداخلها

② Imagine a vertical line cutting through R. in the direction of increasing (y) mark the (y) values where .L. enteres and leaves these are the (y) limits of integration,

$$\int \int_{1-x}^{\sqrt{1-x^2}} dy\,dx$$

*نقطة الدخول تمثل القيمة السفلى لل(y) اما نقطة المغادرة تمثل القيمة العليا (لل y)

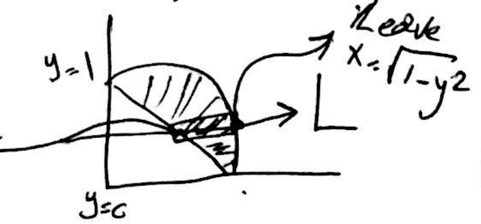


③ choose X-Limits that include all the verticall lines through .R. the integral is :-

$$\int_0^1 \int_{1-x}^{\sqrt{1-x^2}}$$

④ to evaluate the same double integral as an iterated integral with the order of integral the procedures horizantal line insted of vertical line

⑤ the integral $\int_{y=0}^{y=1} \int_{x=1-y}^{x=\sqrt{1-y^2}} dx\,dy$



Ex// Evaluate $\iint_R (4x+2)\,dA$ bounded by $y=2x$ & $y=x^2$

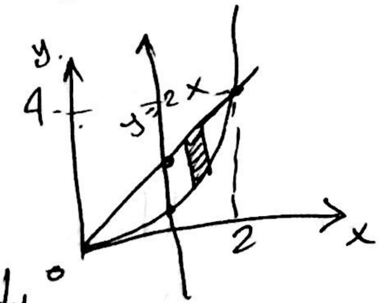
Solution

$y = 2x$

x	y
0	0
1	2
2	4

$y = x^2$

x	y
0	0
1	1
2	4



or// $\int_0^4 \int_{1/2 y}^{\sqrt{y}} (4x+2)\,dx\,dy$

$$= \int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^2 \left[4xy+2y\right]_{x^2}^{2x} dx$$

$$= \int_0^2 (6x^2 - 4x^3 + 4x)\,dx = 2x^3 - x^4 + 2x^2 \Big]_0^2 = 16 - 16 + 8 = 8$$

Ex// evaluate $\iint_R \frac{x}{\sqrt{y}}\,dA$ over the region in the First quadrant bounded by lines $y=x$, $y=2x$, $x=1$, $x=2$

Solution

$$\int_1^2 \int_x^{2x} \left(\frac{x}{\sqrt{y}}\right) dy\,dx = \int_1^2 \left[2x\sqrt{y}\right]_x^{2x} dx = \int_1^2 (2x\sqrt{2x} - 2x\sqrt{x})\,dx$$

$x(y)^{-1/2}$

$\dfrac{2 * \sqrt{y} * 2}{\frac{1}{2}}$

عند الضرب تجمع الاس

$$= \int_1^2 (2x)^{3/2} - (2x)^{3/2}\,dx$$

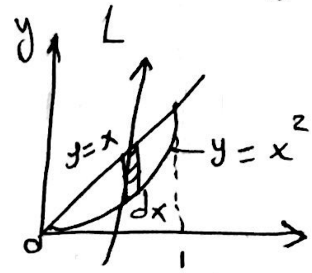
or// $\int_{y=1}^{y=4} \int_{x=\frac{y}{2}}^{x=y} \frac{x}{\sqrt{y}}\,dx\,dy = \int_1^4 \left[\frac{x^2}{2y}\right]_{y/2}^{4} dy = \int_1^4 \left(\frac{y}{2} - \frac{y}{8}\right) dy$

$$= \int_1^4 \left(\frac{3y}{8}\right) dy = \frac{3y^2}{16}\Big]_1^4$$

Areas:- The area of closed bounded plane region R it is the value of the integral.

$$Area = \iint_R dx\,dy$$

Ex// Find the area of region R bounded by $\boxed{y=x}$ and $\boxed{y=x^2}$ in the first quadrant

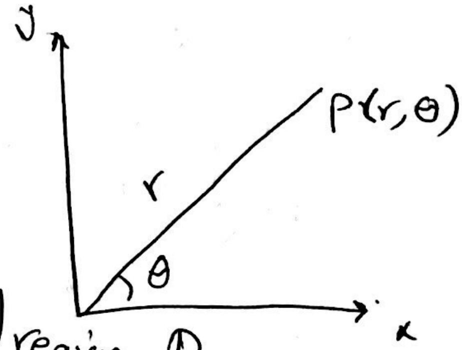


Solution

$$A = \int_0^1 \int_{x^2}^{x} dy\,dx$$

$$= \int_0^1 (x-x^2)\,dx = \frac{x^2}{2} - \frac{x^3}{3}\Big]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ unit Area}$$

- Double integrals in polar form



① $$\iint_R f(r,\theta)\,dA = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=r_1\theta_1}^{r=r_2\theta_2} f(r,\theta)\,r\,dr\,d\theta$$

② The area of a closed and bounded region R in polar coor. is given by

$$A = \iint_R r\,dr\,d\theta$$

Note:- Changing of cartesian to polar coordinates

$$\iint_R f(x,y)\,dy\,dx = \iint_{R_{polar}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$