Equilibrium

3

CHAPTER OUTLINE

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3/1 Introduction

Statics deals primarily with the description of the force conditions necessary and sufficient to maintain the equilibrium of engineering structures. This chapter on equilibrium, therefore, constitutes the most important part of statics, and the procedures developed here form the basis for solving problems in both statics and dynamics. We will make continual use of the concepts developed in Chapter 2 involving forces, moments, couples, and resultants as we apply the principles of equilibrium.

When a body is in equilibrium, the resultant of *all* forces acting on it is zero. Thus, the resultant force **R** and the resultant couple **M** are both zero, and we have the equilibrium equations

$$\mathbf{R} = \Sigma\mathbf{F} = \mathbf{0} \qquad \mathbf{M} = \Sigma\mathbf{M} = \mathbf{0} \tag{3/1}$$

These requirements are both necessary and sufficient conditions for equilibrium.

All physical bodies are three-dimensional, but we can treat many of them as two-dimensional when the forces to which they are subjected act in a single plane or can be projected onto a single plane. When this simplification is not possible, the problem must be treated as three-dimensional. We will follow the arrangement used in Chapter 2, and discuss in Section A the equilibrium of bodies subjected to two-dimensional force systems and in Section B the equilibrium of bodies subjected to three-dimensional force systems.

SECTION A EQUILIBRIUM IN TWO DIMENSIONS

3/2 System Isolation and the Free-Body Diagram

Before we apply Eqs. 3/1, we must define unambiguously the particular body or mechanical system to be analyzed and represent clearly and completely *all* forces acting *on* the body. Omission of a force which acts *on* the body in question, or inclusion of a force which does not act *on* the body, will give erroneous results.

A *mechanical system* is defined as a body or group of bodies which can be conceptually isolated from all other bodies. A system may be a single body or a combination of connected bodies. The bodies may be rigid or nonrigid. The system may also be an identifiable fluid mass, either liquid or gas, or a combination of fluids and solids. In statics we study primarily forces which act on rigid bodies at rest, although we also study forces acting on fluids in equilibrium.

Once we decide which body or combination of bodies to analyze, we then treat this body or combination as a single body *isolated* from all surrounding bodies. This isolation is accomplished by means of the ***free-body diagram***, which is a diagrammatic representation of the isolated system treated as a single body. The diagram shows all forces applied to the system by mechanical contact with other bodies, which are imagined to be removed. If appreciable body forces are present, such as gravitational or magnetic attraction, then these forces must also be shown on the free-body diagram of the isolated system. Only after such a diagram has been carefully drawn should the equilibrium equations be written. Because of its critical importance, we emphasize here that

> **the free-body diagram is the most important single step in the solution of problems in mechanics.**

Before attempting to draw a free-body diagram, we must recall the basic characteristics of force. These characteristics were described in Art. 2/2, with primary attention focused on the vector properties of force. Forces can be applied either by direct physical contact or by remote action. Forces can be either internal or external to the system under consideration. Application of force is accompanied by reactive force, and both applied and reactive forces may be either concentrated or distributed. The principle of transmissibility permits the treatment of force as a sliding vector as far as its external effects on a rigid body are concerned.

We will now use these force characteristics to develop conceptual models of isolated mechanical systems. These models enable us to write the appropriate equations of equilibrium, which can then be analyzed.

Modeling the Action of Forces

Figure 3/1 shows the common types of force application on mechanical systems for analysis in two dimensions. Each example shows the force exerted *on* the body to be *isolated*, *by* the body to be *removed*. Newton's third law, which notes the existence of an equal and opposite reaction to every action, must be carefully observed. The force exerted *on* the body in question *by* a contacting or supporting member is always in the sense to oppose the movement of the isolated body which would occur if the contacting or supporting body were removed.

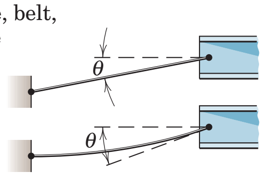
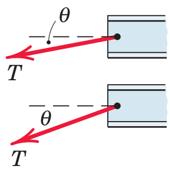
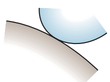
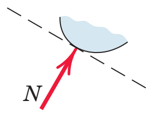
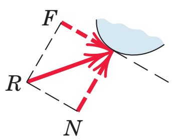
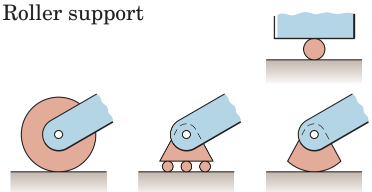
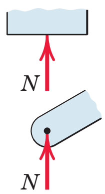
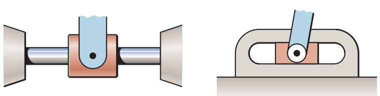
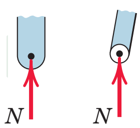
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS		
Type of Contact and Force Origin	Action on Body to Be Isolated	
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 		Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 		Contact force is compressive and is normal to the surface.
3. Rough surfaces 		Rough surfaces are capable of supporting a tangential component *F* (frictional force) as well as a normal component *N* of the resultant contact force *R*.
4. Roller support 		Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 		Collar or slider free to move along smooth guides; can support force normal to guide only.

Figure 3/1

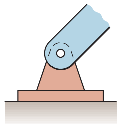
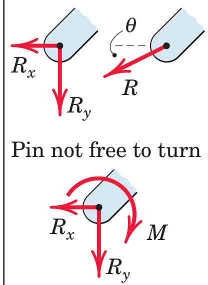
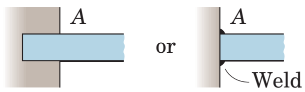
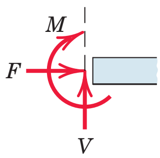
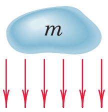
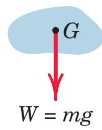
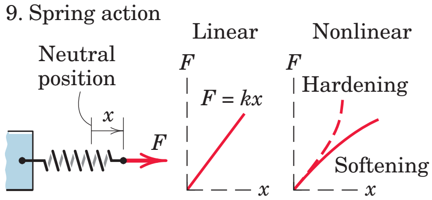
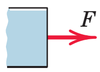
MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)	
Type of Contact and Force Origin	Action on Body to Be Isolated
6. Pin connection 	 A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

Figure 3/1, continued

In Fig. 3/1, Example 1 depicts the action of a flexible cable, belt, rope, or chain on the body to which it is attached. Because of its flexibility, a rope or cable is unable to offer any resistance to bending, shear, or compression and therefore exerts only a tension force in a direction tangent to the cable at its point of attachment. The force exerted *by* the cable *on* the body to which it is attached is always *away* from the body. When the tension T is large compared with the weight of the cable, we may assume that the cable forms a straight line. When the cable weight is not negligible compared with its tension, the sag of the cable becomes important, and the tension in the cable changes direction and magnitude along its length.

When the smooth surfaces of two bodies are in contact, as in Example 2, the force exerted by one on the other is *normal* to the tangent to the surfaces and is compressive. Although no actual surfaces are perfectly smooth, we can assume this to be so for practical purposes in many instances.

When mating surfaces of contacting bodies are rough, as in Example 3, the force of contact is not necessarily normal to the tangent to the surfaces, but may be resolved into a *tangential* or *frictional component* F and a *normal component* N.

Example 4 illustrates a number of forms of mechanical support which effectively eliminate tangential friction forces. In these cases the net reaction is normal to the supporting surface.

Example 5 shows the action of a smooth guide on the body it supports. There cannot be any resistance parallel to the guide.

Example 6 illustrates the action of a pin connection. Such a connection can support force in any direction normal to the axis of the pin. We usually represent this action in terms of two rectangular components. The correct sense of these components in a specific problem depends on how the member is loaded. When not otherwise initially known, the sense is arbitrarily assigned and the equilibrium equations are then written. If the solution of these equations yields a positive algebraic sign for the force component, the assigned sense is correct. A negative sign indicates the sense is opposite to that initially assigned.

If the joint is free to turn about the pin, the connection can support only the force R. If the joint is not free to turn, the connection can also support a resisting couple M. The sense of M is arbitrarily shown here, but the true sense depends on how the member is loaded.

Example 7 shows the resultants of the rather complex distribution of force over the cross section of a slender bar or beam at a built-in or fixed support. The sense of the reactions F and V and the bending couple M in a given problem depends, of course, on how the member is loaded.

One of the most common forces is that due to gravitational attraction, Example 8. This force affects all elements of mass in a body and is, therefore, distributed throughout it. The resultant of the gravitational forces on all elements is the weight $W = mg$ of the body, which passes through the center of mass G and is directed toward the center of the earth for earthbound structures. The location of G is frequently obvious from the geometry of the body, particularly where there is symmetry. When the location is not readily apparent, it must be determined by experiment or calculations.

Similar remarks apply to the remote action of magnetic and electric forces. These forces of remote action have the same overall effect on a rigid body as forces of equal magnitude and direction applied by direct external contact.

Example 9 illustrates the action of a *linear* elastic spring and of a *nonlinear* spring with either hardening or softening characteristics. The force exerted by a linear spring, in tension or compression, is given by $F = kx$, where k is the *stiffness* of the spring and x is its deformation measured from the neutral or undeformed position.

The representations in Fig. 3/1 are *not* free-body diagrams, but are merely elements used to construct free-body diagrams. Study these nine conditions and identify them in the problem work so that you can draw the correct free-body diagrams.

This apparatus is designed to hold a car body in equilibrium for a wide range of orientations during vehicle production.

KEY CONCEPTS

Construction of Free-Body Diagrams

The full procedure for drawing a free-body diagram which isolates a body or system consists of the following steps.

Step 1. Decide which system to isolate. The system chosen should usually involve one or more of the desired unknown quantities.

Step 2. Next isolate the chosen system by drawing a diagram which represents its *complete external boundary*. This boundary defines the isolation of the system from *all* other attracting or contacting bodies, which are considered removed. This step is often the most crucial of all. Make certain that you have *completely isolated* the system before proceeding with the next step.

Step 3. Identify all forces which act *on* the isolated system as applied *by* the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system. Make a systematic traverse of the entire boundary to identify all contact forces. Include body forces such as weights, where appreciable. Represent all known forces by vector arrows, each with its proper magnitude, direction, and sense indicated. Each unknown force should be represented by a vector arrow with the unknown magnitude or direction indicated by symbol. If the sense of the vector is also unknown, you must arbitrarily assign a sense. The subsequent calculations with the equilibrium equations will yield a positive quantity if the correct sense was assumed and a negative quantity if the incorrect sense was assumed. It is necessary to be *consistent* with the assigned characteristics of unknown forces throughout all of the calculations. If you are consistent, the solution of the equilibrium equations will reveal the correct senses.

Step 4. Show the choice of coordinate axes directly on the diagram. Pertinent dimensions may also be represented for convenience. Note, however, that the free-body diagram serves the purpose of focusing attention on the action of the external forces, and therefore the diagram should not be cluttered with excessive extraneous information. Clearly distinguish force arrows from arrows representing quantities other than forces. For this purpose a colored pencil may be used.

Completion of the foregoing four steps will produce a correct free-body diagram to use in applying the governing equations, both in statics and in dynamics. Be careful not to omit from the free-body diagram certain forces which may not appear at first glance to be needed in the calculations. It is only through *complete* isolation and a systematic representation of *all* external forces that a reliable accounting of the effects of all applied and reactive forces can be made. Very often a force which at first glance may not appear to influence a desired result does indeed have an influence. Thus, the only safe procedure is to include on the free-body diagram all forces whose magnitudes are not obviously negligible.

The free-body method is extremely important in mechanics because it ensures an accurate definition of a mechanical system and focuses attention on the exact meaning and application of the force laws of statics and dynamics. Review the foregoing four steps for constructing a free-body diagram while studying the sample free-body diagrams shown in Fig. 3/2 and the Sample Problems which appear at the end of the next article.

Examples of Free-Body Diagrams

Figure 3/2 gives four examples of mechanisms and structures together with their correct free-body diagrams. Dimensions and magnitudes are omitted for clarity. In each case we treat the entire system as

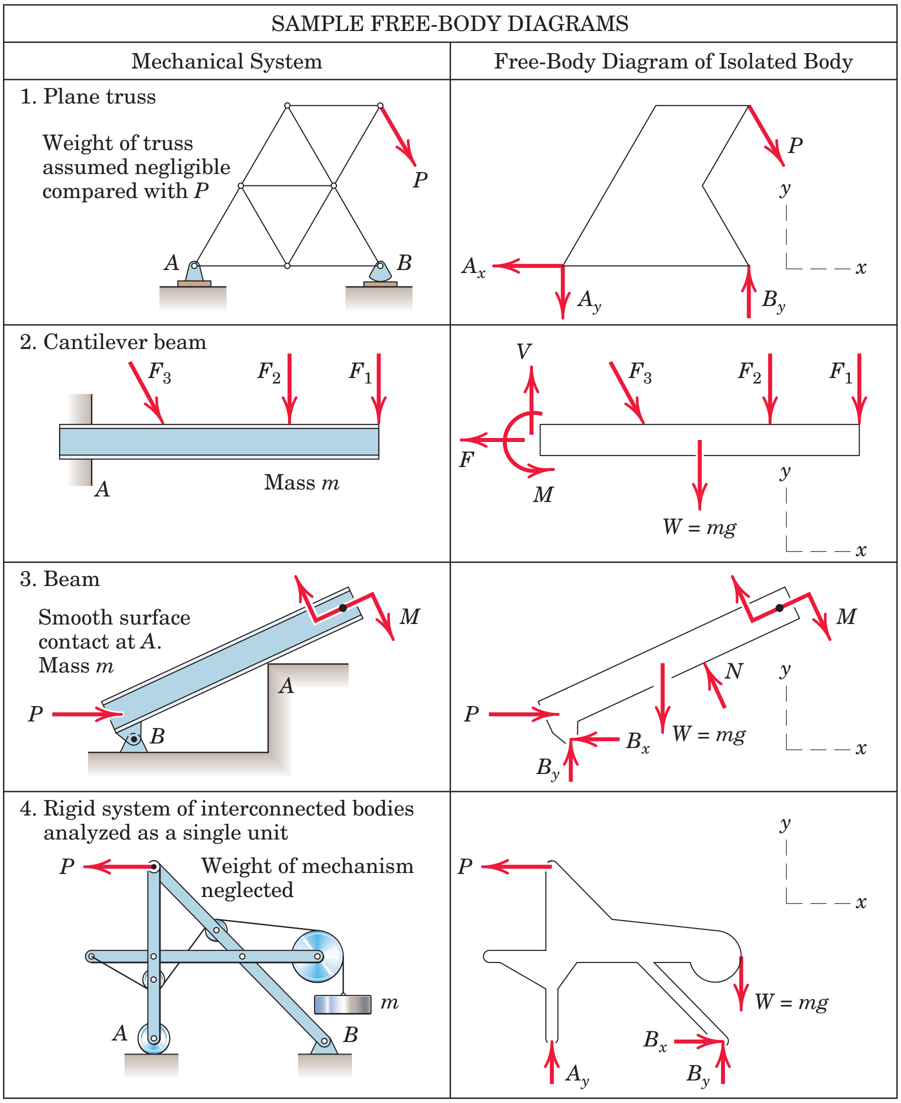


Figure 3/2

a single body, so that the internal forces are not shown. The characteristics of the various types of contact forces illustrated in Fig. 3/1 are used in the four examples as they apply.

In Example 1 the truss is composed of structural elements which, taken all together, constitute a rigid framework. Thus, we may remove the entire truss from its supporting foundation and treat it as a single rigid body. In addition to the applied external load P, the free-body diagram must include the reactions on the truss at A and B. The rocker at B can support a vertical force only, and this force is transmitted to the structure at B (Example 4 of Fig. 3/1). The pin connection at A (Example 6 of Fig. 3/1) is capable of supplying both a horizontal and a vertical force component to the truss. If the total weight of the truss members is appreciable compared with P and the forces at A and B, then the weights of the members must be included on the free-body diagram as external forces.

In this relatively simple example it is clear that the vertical component A_y must be directed down to prevent the truss from rotating clockwise about B. Also, the horizontal component A_x will be to the left to keep the truss from moving to the right under the influence of the horizontal component of P. Thus, in constructing the free-body diagram for this simple truss, we can easily perceive the correct sense of each of the components of force exerted *on* the truss *by* the foundation at A and can, therefore, represent its correct physical sense on the diagram. When the correct physical sense of a force or its component is not easily recognized by direct observation, it must be assigned arbitrarily, and the correctness of or error in the assignment is determined by the algebraic sign of its calculated value.

In Example 2 the cantilever beam is secured to the wall and subjected to three applied loads. When we isolate that part of the beam to the right of the section at A, we must include the reactive forces applied *to* the beam *by* the wall. The resultants of these reactive forces are shown acting on the section of the beam (Example 7 of Fig. 3/1). A vertical force V to counteract the excess of downward applied force is shown, and a tension F to balance the excess of applied force to the right must also be included. Then, to prevent the beam from rotating about A, a counterclockwise couple M is also required. The weight mg of the beam must be represented through the mass center (Example 8 of Fig. 3/1).

In the free-body diagram of Example 2, we have represented the somewhat complex system of forces which actually act on the cut section of the beam by the equivalent force–couple system in which the force is broken down into its vertical component V (shear force) and its horizontal component F (tensile force). The couple M is the bending moment in the beam. The free-body diagram is now complete and shows the beam in equilibrium under the action of six forces and one couple.

In Example 3 the weight $W = mg$ is shown acting through the center of mass of the beam, whose location is assumed known (Example 8 of Fig. 3/1). The force exerted by the corner A on the beam is normal to the smooth surface of the beam (Example 2 of Fig. 3/1). To perceive this action more clearly, visualize an enlargement of the contact point A, which would appear somewhat rounded, and consider the force exerted by this rounded corner on the straight surface of the beam, which is assumed to be smooth. If the contacting surfaces at the corner were not smooth, a tangential frictional component of force could exist. In addition to the applied force P and couple M, there is the pin connection at B, which exerts both an x- and a y-component of force on the beam. The positive senses of these components are assigned arbitrarily.

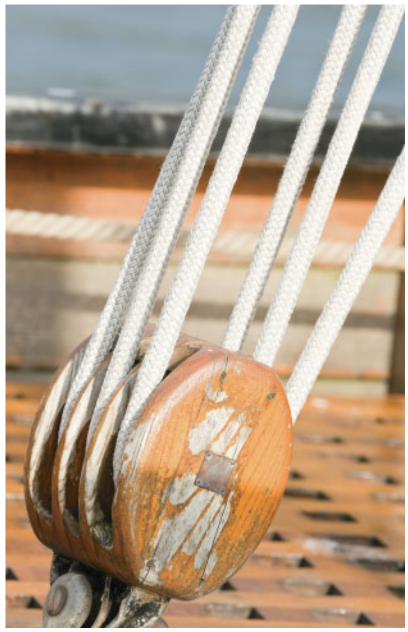
In Example 4 the free-body diagram of the entire isolated mechanism contains three unknown forces if the loads mg and P are known. Any one of many internal configurations for securing the cable leading from the mass m would be possible without affecting the external response of the mechanism as a whole, and this fact is brought out by the free-body diagram. This hypothetical example is used to show that the forces internal to a rigid assembly of members do not influence the values of the external reactions.

We use the free-body diagram in writing the equilibrium equations, which are discussed in the next article. When these equations are solved, some of the calculated force magnitudes may be zero. This would indicate that the assumed force does not exist. In Example 1 of Fig. 3/2, any of the reactions A_x, A_y, or B_y can be zero for specific values of the truss geometry and of the magnitude, direction, and sense of the applied load P. A zero reaction force is often difficult to identify by inspection, but can be determined by solving the equilibrium equations.

Similar comments apply to calculated force magnitudes which are negative. Such a result indicates that the actual sense is the opposite of the assumed sense. The assumed positive senses of B_x and B_y in Example 3 and B_y in Example 4 are shown on the free-body diagrams. The correctness of these assumptions is proved or disproved according to whether the algebraic signs of the computed forces are plus or minus when the calculations are carried out in an actual problem.

The isolation of the mechanical system under consideration is a crucial step in the formulation of the mathematical model. The most important aspect to the correct construction of the all-important free-body diagram is the clear-cut and unambiguous decision as to what is included and what is excluded. This decision becomes unambiguous only when the boundary of the free-body diagram represents a complete traverse of the body or system of bodies to be isolated, starting at some arbitrary point on the boundary and returning to that same point. The system within this closed boundary is the isolated free body, and all contact forces and all body forces transmitted to the system across the boundary must be accounted for.

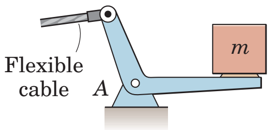
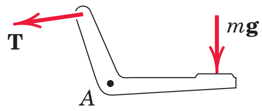
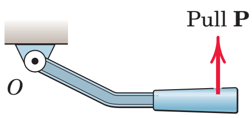
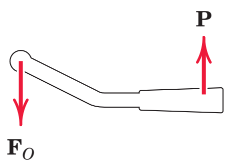
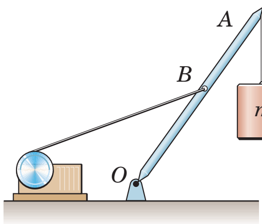
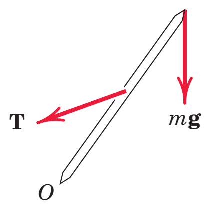
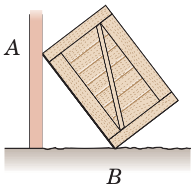
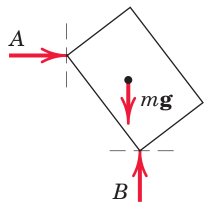
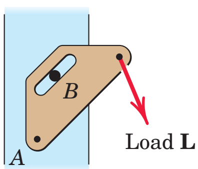
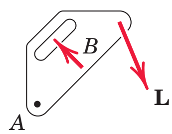
The following exercises provide practice with drawing free-body diagrams. This practice is helpful before using such diagrams in the application of the principles of force equilibrium in the next article.



Complex pulley systems are easily handled with a systematic equilibrium analysis.

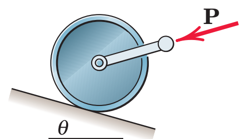
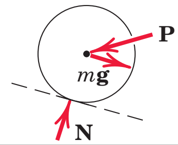
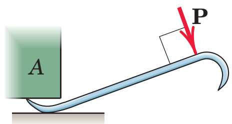
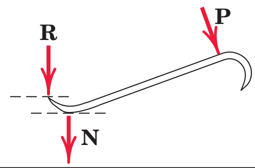
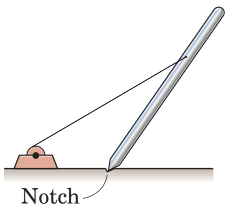
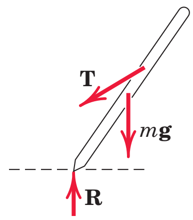
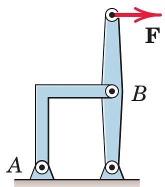
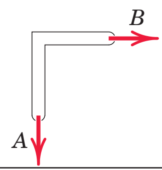
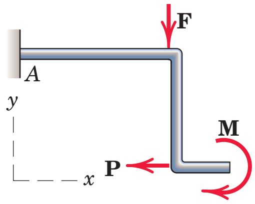
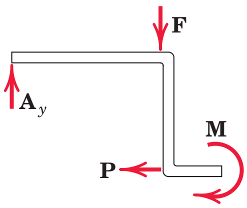
FREE-BODY DIAGRAM EXERCISES

3/A In each of the five following examples, the body to be isolated is shown in the left-hand diagram, and an *incomplete* free-body diagram (FBD) of the isolated body is shown on the right. Add whatever forces are necessary in each case to form a complete free-body diagram. The weights of the bodies are negligible unless otherwise indicated. Dimensions and numerical values are omitted for simplicity.

	Body	Incomplete FBD
1. Bell crank supporting mass m with pin support at A.		
2. Control lever applying torque to shaft at O.		
3. Boom OA, of negligible mass compared with mass m. Boom hinged at O and supported by hoisting cable at B.		
4. Uniform crate of mass m leaning against smooth vertical wall and supported on a rough horizontal surface.		
5. Loaded bracket supported by pin connection at A and fixed pin in smooth slot at B.		

Problem 3/A

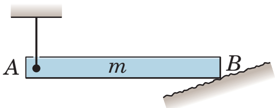
3/B In each of the five following examples, the body to be isolated is shown in the left-hand diagram, and either a *wrong* or an *incomplete* free-body diagram (FBD) is shown on the right. Make whatever changes or additions are necessary in each case to form a correct and complete free-body diagram. The weights of the bodies are negligible unless otherwise indicated. Dimensions and numerical values are omitted for simplicity.

	Body	Wrong or Incomplete FBD
1. Lawn roller of mass m being pushed up incline θ.		
2. Prybar lifting body A having smooth horizontal surface. Bar rests on horizontal rough surface.		
3. Uniform pole of mass m being hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of pole.		
4. Supporting angle bracket for frame; pin joints.		
5. Bent rod welded to support at A and subjected to two forces and couple.		

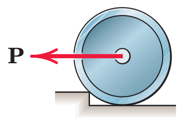
Problem 3/B

3/C Draw a complete and correct free-body diagram of each of the bodies designated in the statements. The weights of the bodies are significant only if the mass is stated. All forces, known and unknown, should be labeled. (*Note*: The sense of some reaction components cannot always be determined without numerical calculation.)

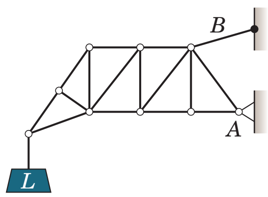
1. Uniform horizontal bar of mass m suspended by vertical cable at A and supported by rough inclined surface at B.



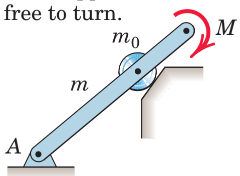
2. Wheel of mass m on verge of being rolled over curb by pull **P**.



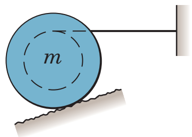
3. Loaded truss supported by pin joint at A and by cable at B.



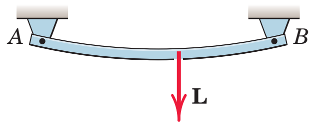
4. Uniform bar of mass m and roller of mass m_0 taken together. Subjected to couple M and supported as shown. Roller is free to turn.



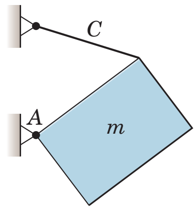
5. Uniform grooved wheel of mass m supported by a rough surface and by action of horizontal cable.



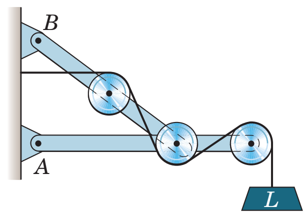
6. Bar, initially horizontal but deflected under load **L**. Pinned to rigid support at each end.



7. Uniform heavy plate of mass m supported in vertical plane by cable C and hinge A.



8. Entire frame, pulleys, and contacting cable to be isolated as a single unit.



Problem 3/C

3/3 Equilibrium Conditions

In Art. 3/1 we defined equilibrium as the condition in which the resultant of all forces and moments acting on a body is zero. Stated in another way, a body is in equilibrium if all forces and moments applied to it are in balance. These requirements are contained in the vector equations of equilibrium, Eqs. 3/1, which in two dimensions may be written in scalar form as

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0 \tag{3/2}$$

The third equation represents the zero sum of the moments of all forces about any point O on or off the body. Equations 3/2 are the necessary and sufficient conditions for complete equilibrium in two dimensions. They are necessary conditions because, if they are not satisfied, there can be no force or moment balance. They are sufficient because once they are satisfied, there can be no imbalance, and equilibrium is assured.

The equations relating force and acceleration for rigid-body motion are developed in *Vol. 2 Dynamics* from Newton's second law of motion. These equations show that the acceleration of the mass center of a body is proportional to the resultant force $\Sigma\mathbf{F}$ acting on the body. Consequently, if a body moves with constant velocity (zero acceleration), the resultant force on it must be zero, and the body may be treated as in a state of translational equilibrium.

For complete equilibrium in two dimensions, all three of Eqs. 3/2 must hold. However, these conditions are independent requirements, and one may hold without another. Take, for example, a body which slides along a horizontal surface with increasing velocity under the action of applied forces. The force–equilibrium equations will be satisfied in the vertical direction where the acceleration is zero, but not in the horizontal direction. Also, a body, such as a flywheel, which rotates about its fixed mass center with increasing angular speed is not in rotational equilibrium, but the two force–equilibrium equations will be satisfied.

Categories of Equilibrium

Applications of Eqs. 3/2 fall naturally into a number of categories which are easily identified. The categories of force systems acting on bodies in two-dimensional equilibrium are summarized in Fig. 3/3 and are explained further as follows.

Category 1, equilibrium of collinear forces, clearly requires only the one force equation in the direction of the forces (x-direction), since all other equations are automatically satisfied.

Category 2, equilibrium of forces which lie in a plane (x-y plane) and are concurrent at a point O, requires the two force equations only, since the moment sum about O, that is, about a z-axis through O, is necessarily zero. Included in this category is the case of the equilibrium of a particle.

Category 3, equilibrium of parallel forces in a plane, requires the one force equation in the direction of the forces (x-direction) and one moment equation about an axis (z-axis) normal to the plane of the forces.

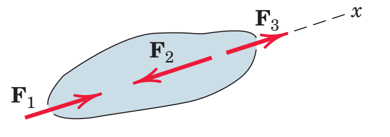
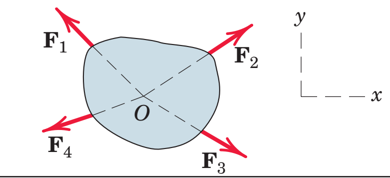
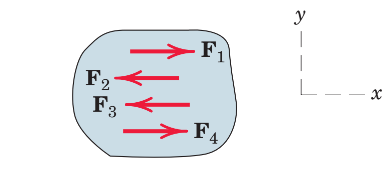
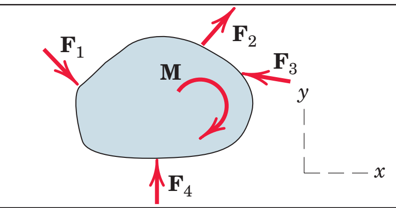
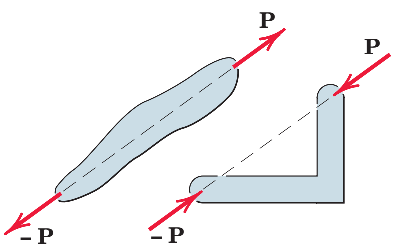
CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\Sigma M_z = 0$ $\Sigma F_y = 0$

Figure 3/3

Category 4, equilibrium of a general system of forces in a plane (x-y), requires the two force equations in the plane and one moment equation about an axis (z-axis) normal to the plane.



Two-force members

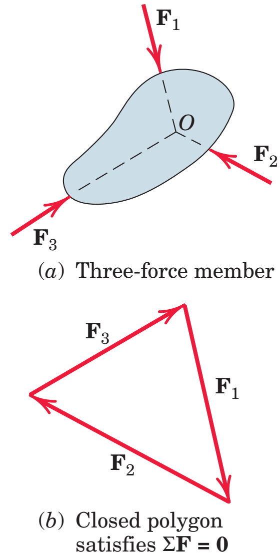
Figure 3/4

Two- and Three-Force Members

You should be alert to two frequently occurring equilibrium situations. The first situation is the equilibrium of a body under the action of two forces only. Two examples are shown in Fig. 3/4, and we see that for such a *two-force member* to be in equilibrium, the forces must be *equal*, *opposite*, and *collinear*. The shape of the member does not affect this simple requirement. In the illustrations cited, we consider the weights of the members to be negligible compared with the applied forces.

The second situation is a *three-force member*, which is a body under the action of three forces, Fig. 3/5a. We see that equilibrium requires the lines of action of the three forces to be *concurrent*. If they were not concurrent, then one of the forces would exert a resultant moment about the point of intersection of the other two, which would violate the requirement of zero moment about every point. The only exception occurs when the three forces are parallel. In this case we may consider the point of concurrency to be at infinity.

The principle of the concurrency of three forces in equilibrium is of considerable use in carrying out a graphical solution of the force equations. In this case the polygon of forces is drawn and made to close, as shown in Fig. 3/5*b*. Frequently, a body in equilibrium under the action of more than three forces may be reduced to a three-force member by a combination of two or more of the known forces.



(*a*) Three-force member

(*b*) Closed polygon satisfies $\Sigma\mathbf{F} = \mathbf{0}$

Figure 3/5

Alternative Equilibrium Equations

In addition to Eqs. 3/2, there are two other ways to express the general conditions for the equilibrium of forces in two dimensions. The first way is illustrated in Fig. 3/6, parts (*a*) and (*b*). For the body shown in Fig. 3/6*a*, if $\Sigma M_A = 0$, then the resultant, if it still exists, cannot be a couple, but must be a force **R** passing through A. If now the equation $\Sigma F_x = 0$ holds, where the x-direction is arbitrary, it follows from Fig. 3/6*b* that the resultant force **R**, if it still exists, not only must pass through A, but also must be perpendicular to the x-direction as shown. Now, if $\Sigma M_B = 0$, where B is any point such that the line AB is not perpendicular to the x-direction, we see that **R** must be zero, and thus the body is in equilibrium. Therefore, an alternative set of equilibrium equations is

$$\Sigma F_x = 0 \qquad \Sigma M_A = 0 \qquad \Sigma M_B = 0$$

where the two points A and B must not lie on a line perpendicular to the x-direction.

A third formulation of the equilibrium conditions may be made for a coplanar force system. This is illustrated in Fig. 3/6, parts (*c*) and (*d*). Again, if $\Sigma M_A = 0$ for any body such as that shown in Fig. 3/6*c*, the resultant, if any, must be a force **R** through A. In addition, if $\Sigma M_B = 0$, the resultant, if one still exists, must pass through B as shown in Fig. 3/6*d*. Such a force cannot exist, however, if $\Sigma M_C = 0$, where C is not

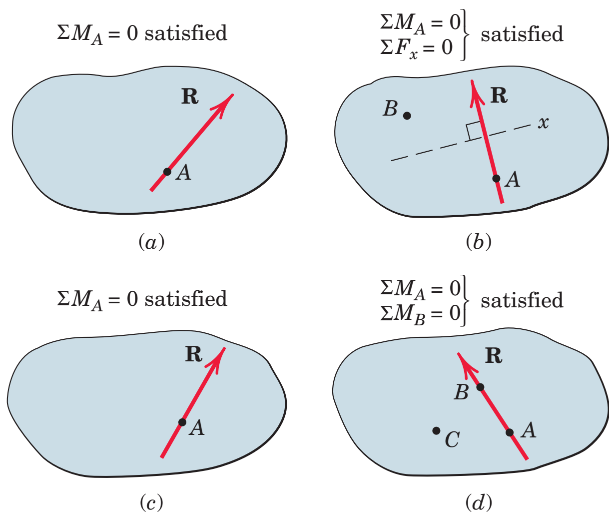


Figure 3/6

collinear with A and B. Thus, we may write the equations of equilibrium as

$$\Sigma M_A = 0 \qquad \Sigma M_B = 0 \qquad \Sigma M_C = 0$$

where A, B, and C are any three points not on the same straight line.

When equilibrium equations are written which are not independent, redundant information is obtained, and a correct solution of the equations will yield $0 = 0$. For example, for a general problem in two dimensions with three unknowns, three moment equations written about three points which lie on the same straight line are not independent. Such equations will contain duplicated information, and solution of two of them can at best determine two of the unknowns, with the third equation merely verifying the identity $0 = 0$.

Constraints and Statical Determinacy

The equilibrium equations developed in this article are both necessary and sufficient conditions to establish the equilibrium of a body. However, they do not necessarily provide all the information required to calculate all the unknown forces which may act on a body in equilibrium. Whether the equations are adequate to determine all the unknowns depends on the characteristics of the constraints against possible movement of the body provided by its supports. By *constraint* we mean the restriction of movement.

In Example 4 of Fig. 3/1 the roller, ball, and rocker provide constraint normal to the surface of contact, but none tangent to the surface. Thus, a tangential force cannot be supported. For the collar and slider of Example 5, constraint exists only normal to the guide. In Example 6 the fixed-pin connection provides constraint in both directions, but offers no resistance to rotation about the pin unless the pin is not free to turn. The fixed support of Example 7, however, offers constraint against rotation as well as lateral movement.

If the rocker which supports the truss of Example 1 in Fig. 3/2 were replaced by a pin joint, as at A, there would be one additional constraint beyond those required to support an equilibrium configuration with no freedom of movement. The three scalar conditions of equilibrium, Eqs. 3/2, would not provide sufficient information to determine all four unknowns, since A_x and B_x could not be solved for separately; only their sum could be determined. These two components of force would be dependent on the deformation of the members of the truss as influenced by their corresponding stiffness properties. The horizontal reactions A_x and B_x would also depend on any initial deformation required to fit the dimensions of the structure to those of the foundation between A and B. Thus, we cannot determine A_x and B_x by a rigid-body analysis.

Again referring to Fig. 3/2, we see that if the pin B in Example 3 were not free to turn, the support could transmit a couple to the beam through the pin. Therefore, there would be four unknown supporting reactions acting on the beam—namely, the force at A, the two components of force at B, and the couple at B. Consequently the three independent scalar equations of equilibrium would not provide enough information to compute all four unknowns.

A rigid body, or rigid combination of elements treated as a single body, which possesses more external supports or constraints than are necessary to maintain an equilibrium position is called *statically indeterminate*. Supports which can be removed without destroying the equilibrium condition of the body are said to be *redundant*. The number of redundant supporting elements present corresponds to the *degree of statical indeterminacy* and equals the total number of unknown external forces, minus the number of available independent equations of equilibrium. On the other hand, bodies which are supported by the minimum number of constraints necessary to ensure an equilibrium configuration are called *statically determinate*, and for such bodies the equilibrium equations are sufficient to determine the unknown external forces.

The problems on equilibrium in this article and throughout *Vol. 1 Statics* are generally restricted to statically determinate bodies where the constraints are just sufficient to ensure a stable equilibrium configuration and where the unknown supporting forces can be completely determined by the available independent equations of equilibrium.

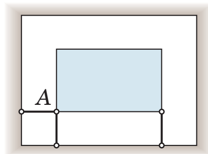
We must be aware of the nature of the constraints before we attempt to solve an equilibrium problem. A body can be recognized as statically indeterminate when there are more unknown external reactions than there are available independent equilibrium equations for the force system involved. It is always well to count the number of unknown variables on a given body and to be certain that an equal number of independent equations can be written; otherwise, effort might be wasted in attempting an impossible solution with the aid of the equilibrium equations only. The unknown variables may be forces, couples, distances, or angles.

Adequacy of Constraints

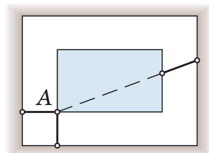
In discussing the relationship between constraints and equilibrium, we should look further at the question of the adequacy of constraints. The existence of three constraints for a two-dimensional problem does not always guarantee a stable equilibrium configuration. Figure 3/7 shows four different types of constraints. In part *a* of the figure, point *A* of the rigid body is fixed by the two links and cannot move, and the third link prevents any rotation about *A*. Thus, this body is *completely fixed* with three *adequate* (*proper*) *constraints*.

In part *b* of the figure, the third link is positioned so that the force transmitted by it passes through point *A* where the other two constraint forces act. Thus, this configuration of constraints can offer no initial resistance to rotation about *A*, which would occur when external loads were applied to the body. We conclude, therefore, that this body is *incompletely fixed* under *partial constraints*.

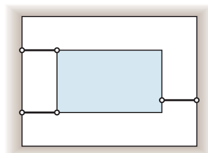
The configuration in part *c* of the figure gives us a similar condition of incomplete fixity because the three parallel links could offer no initial resistance to a small vertical movement of the body as a result of external loads applied to it in this direction. The constraints in these two examples are often termed *improper*.



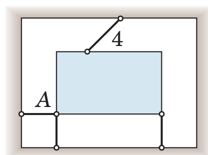
(*a*) Complete fixity
Adequate constraints



(*b*) Incomplete fixity
Partial constraints



(*c*) Incomplete fixity
Partial constraints



(*d*) Excessive fixity
Redundant constraint

Figure 3/7

In part *d* of Fig. 3/7 we have a condition of complete fixity, with link 4 acting as a fourth constraint which is unnecessary to maintain a fixed position. Link 4, then, is a *redundant constraint*, and the body is statically indeterminate.

As in the four examples of Fig. 3/7, it is generally possible by direct observation to conclude whether the constraints on a body in two-dimensional equilibrium are adequate (proper), partial (improper), or redundant. As indicated previously, the vast majority of problems in this book are statically determinate with adequate (proper) constraints.

KEY CONCEPTS

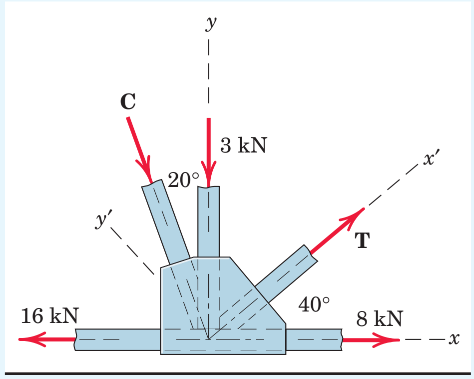
Approach to Solving Problems

The sample problems at the end of this article illustrate the application of free-body diagrams and the equations of equilibrium to typical statics problems. These solutions should be studied thoroughly. In the problem work of this chapter and throughout mechanics, it is important to develop a logical and systematic approach which includes the following steps:

1. Identify clearly the quantities which are known and unknown.
2. Make an unambiguous choice of the body (or system of connected bodies treated as a single body) to be isolated and draw its complete free-body diagram, labeling all external known and unknown but identifiable forces and couples which act on it.
3. Choose a convenient set of reference axes, always using right-handed axes when vector cross products are employed. Choose moment centers with a view to simplifying the calculations. Generally the best choice is one through which as many unknown forces pass as possible. Simultaneous solutions of equilibrium equations are frequently necessary, but can be minimized or avoided by a careful choice of reference axes and moment centers.
4. Identify and state the applicable force and moment principles or equations which govern the equilibrium conditions of the problem. In the following sample problems these relations are shown in brackets and precede each major calculation.
5. Match the number of independent equations with the number of unknowns in each problem.
6. Carry out the solution and check the results. In many problems engineering judgment can be developed by first making a reasonable guess or estimate of the result prior to the calculation and then comparing the estimate with the calculated value.

SAMPLE PROBLEM 3/1

Determine the magnitudes of the forces **C** and **T**, which, along with the other three forces shown, act on the bridge-truss joint.



Solution. The given sketch constitutes the free-body diagram of the isolated
1 section of the joint in question and shows the five forces which are in equilibrium.

Solution 1 (scalar algebra). For the x-y axes as shown we have

$$[\Sigma F_x = 0] \qquad 8 + T\cos 40° + C\sin 20° - 16 = 0$$

$$0.766T + 0.342C = 8 \qquad (a)$$

$$[\Sigma F_y = 0] \qquad T\sin 40° - C\cos 20° - 3 = 0$$

$$0.643T - 0.940C = 3 \qquad (b)$$

Simultaneous solution of Eqs. (a) and (b) produces

$$T = 9.09 \text{ kN} \qquad C = 3.03 \text{ kN} \qquad \textit{Ans.}$$

Solution II (scalar algebra). To avoid a simultaneous solution, we may use axes
2 x'-y' with the first summation in the y'-direction to eliminate reference to T. Thus,

$$[\Sigma F_{y'} = 0] \qquad -C\cos 20° - 3\cos 40° - 8\sin 40° + 16\sin 40° = 0$$

$$C = 3.03 \text{ kN} \qquad \textit{Ans.}$$

$$[\Sigma F_{x'} = 0] \qquad T + 8\cos 40° - 16\cos 40° - 3\sin 40° - 3.03\sin 20° = 0$$

$$T = 9.09 \text{ kN} \qquad \textit{Ans.}$$

Solution III (vector algebra). With unit vectors **i** and **j** in the x- and y-directions, the zero summation of forces for equilibrium yields the vector equation

$$[\Sigma \mathbf{F} = \mathbf{0}] \qquad 8\mathbf{i} + (T\cos 40°)\mathbf{i} + (T\sin 40°)\mathbf{j} - 3\mathbf{j} + (C\sin 20°)\mathbf{i} - (C\cos 20°)\mathbf{j} - 16\mathbf{i} = \mathbf{0}$$

Equating the coefficients of the **i**- and **j**-terms to zero gives

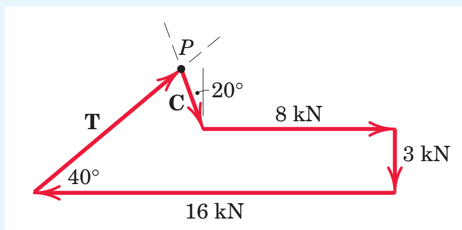
$$8 + T\cos 40° + C\sin 20° - 16 = 0$$

$$T\sin 40° - 3 - C\cos 20° = 0$$

which are the same, of course, as Eqs. (a) and (b), which we solved above.

Solution IV (geometric). The polygon representing the zero vector sum of the five forces is shown. Equations (a) and (b) are seen immediately to give the projections of the vectors onto the x- and y-directions. Similarly, projections onto the x'- and y'-directions give the alternative equations in Solution II.

A graphical solution is easily obtained. The known vectors are laid off head-
3 to-tail to some convenient scale, and the directions of **T** and **C** are then drawn to close the polygon. The resulting intersection at point P completes the solution, thus enabling us to measure the magnitudes of **T** and **C** directly from the drawing to whatever degree of accuracy we incorporate in the construction.



Helpful Hints

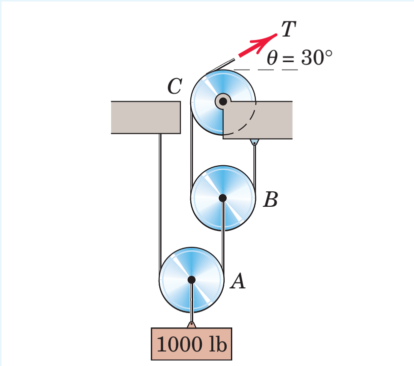
1 Since this is a problem of concurrent forces, no moment equation is necessary.

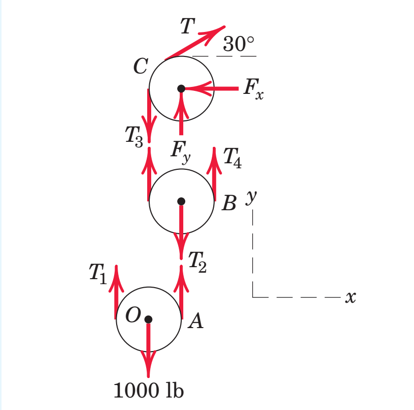
2 The selection of reference axes to facilitate computation is always an important consideration. Alternatively in this example we could take a set of axes along and normal to the direction of **C** and employ a force summation normal to **C** to eliminate it.

3 The known vectors may be added in any order desired, but they must be added before the unknown vectors.

SAMPLE PROBLEM 3/2

Calculate the tension T in the cable which supports the 1000-lb load with the pulley arrangement shown. Each pulley is free to rotate about its bearing, and the weights of all parts are small compared with the load. Find the magnitude of the total force on the bearing of pulley C.





Solution. The free-body diagram of each pulley is drawn in its relative position to the others. We begin with pulley A, which includes the only known force. With the unspecified pulley radius designated by r, the equilibrium of moments about its center O and the equilibrium of forces in the vertical direction require

① $[\Sigma M_O = 0]$ $\qquad T_1 r - T_2 r = 0 \qquad T_1 = T_2$

$[\Sigma F_y = 0]$ $\qquad T_1 + T_2 - 1000 = 0 \qquad 2T_1 = 1000 \qquad T_1 = T_2 = 500 \text{ lb}$

From the example of pulley A we may write the equilibrium of forces on pulley B by inspection as

$$T_3 = T_4 = T_2/2 = 250 \text{ lb}$$

For pulley C the angle $\theta = 30°$ in no way affects the moment of T about the center of the pulley, so that moment equilibrium requires

$$T = T_3 \qquad \text{or} \qquad T = 250 \text{ lb} \qquad \textit{Ans.}$$

Equilibrium of the pulley in the x- and y-directions requires

$[\Sigma F_x = 0]$ $\qquad 250 \cos 30° - F_x = 0 \qquad F_x = 217 \text{ lb}$

$[\Sigma F_y = 0]$ $\qquad F_y + 250 \sin 30° - 250 = 0 \qquad F_y = 125 \text{ lb}$

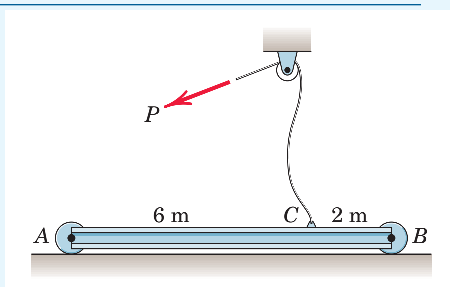
$[F = \sqrt{F_x^2 + F_y^2}]$ $\qquad F = \sqrt{(217)^2 + (125)^2} = 250 \text{ lb} \qquad$ *Ans.*

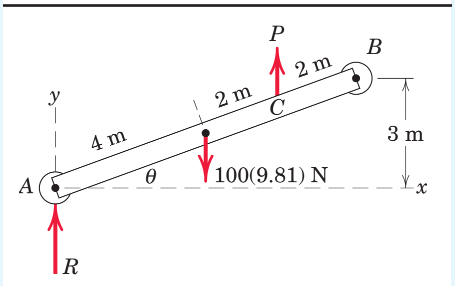
Helpful Hint

① Clearly the radius r does not influence the results. Once we have analyzed a simple pulley, the results should be perfectly clear by inspection.

SAMPLE PROBLEM 3/3

The uniform 100-kg I-beam is supported initially by its end rollers on the horizontal surface at A and B. By means of the cable at C it is desired to elevate end B to a position 3 m above end A. Determine the required tension P, the reaction at A, and the angle θ made by the beam with the horizontal in the elevated position.





Solution. In constructing the free-body diagram, we note that the reaction on the roller at A and the weight are vertical forces. Consequently, in the absence of other horizontal forces, P must also be vertical. From Sample Problem 3/2 we see immediately that the tension P in the cable equals the tension P applied to the beam at C.

Moment equilibrium about A eliminates force R and gives

① $[\Sigma M_A = 0]$ $\qquad P(6 \cos \theta) - 981(4 \cos \theta) = 0 \qquad P = 654 \text{ N} \qquad$ *Ans.*

Equilibrium of vertical forces requires

$[\Sigma F_y = 0]$ $\qquad 654 + R - 981 = 0 \qquad R = 327 \text{ N} \qquad$ *Ans.*

The angle θ depends only on the specified geometry and is

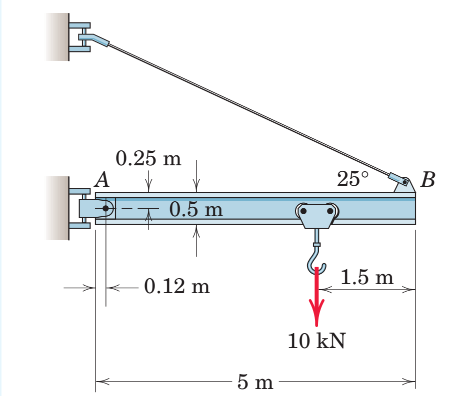
$$\sin \theta = 3/8 \qquad \theta = 22.0° \qquad \textit{Ans.}$$

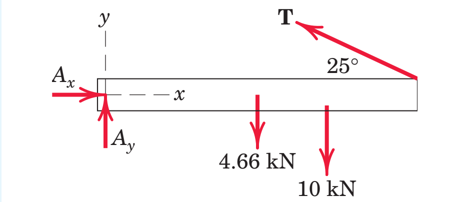
Helpful Hint

① Clearly the equilibrium of this parallel force system is independent of θ.

SAMPLE PROBLEM 3/4

Determine the magnitude T of the tension in the supporting cable and the magnitude of the force on the pin at A for the jib crane shown. The beam AB is a standard 0.5-m I-beam with a mass of 95 kg per meter of length.





Free-body diagram

Algebraic solution. The system is symmetrical about the vertical x-y plane through the center of the beam, so the problem may be analyzed as the equilibrium of a coplanar force system. The free-body diagram of the beam is shown in the figure with the pin reaction at A represented in terms of its two rectangular components. The weight of the beam is $95(10^{-3})(5)9.81 = 4.66$ kN and acts through its center. Note that there are three unknowns A_x, A_y, and T, which may be found from the three equations of equilibrium. We begin with a moment equation about A, which eliminates two of the three unknowns from the equation. ① In applying the moment equation about A, it is simpler to consider the moments of the x- and y-components of $\mathbf{T}$ than it is to compute the perpendicular distance from $\mathbf{T}$ to A. Hence, with the counterclockwise sense as positive we write

② $[\Sigma M_A = 0]$ $$(T\cos 25°)0.25 + (T\sin 25°)(5 - 0.12) - 10(5 - 1.5 - 0.12) - 4.66(2.5 - 0.12) = 0$$

from which $T = 19.61$ kN *Ans.*

Equating the sums of forces in the x- and y-directions to zero gives

$[\Sigma F_x = 0]$ $A_x - 19.61\cos 25° = 0$ $A_x = 17.77$ kN

$[\Sigma F_y = 0]$ $A_y + 19.61\sin 25° - 4.66 - 10 = 0$ $A_y = 6.37$ kN

③ $[A = \sqrt{A_x^2 + A_y^2}]$ $A = \sqrt{(17.77)^2 + (6.37)^2} = 18.88$ kN *Ans.*

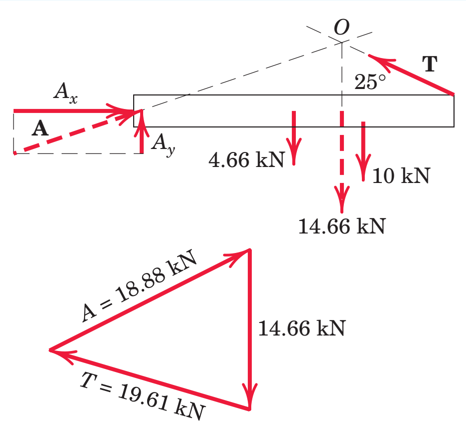
Graphical solution. The principle that three forces in equilibrium must be concurrent is utilized for a graphical solution by combining the two known vertical forces of 4.66 and 10 kN into a single 14.66-kN force, located as shown on the modified free-body diagram of the beam in the lower figure. The position of this resultant load may easily be determined graphically or algebraically. The intersection of the 14.66-kN force with the line of action of the unknown tension $\mathbf{T}$ defines the point of concurrency O through which the pin reaction $\mathbf{A}$ must pass. The unknown magnitudes of $\mathbf{T}$ and $\mathbf{A}$ may now be found by adding the forces head-to-tail to form the closed equilibrium polygon of forces, thus satisfying their zero vector sum. After the known vertical load is laid off to a convenient scale, as shown in the lower part of the figure, a line representing the given direction of the tension $\mathbf{T}$ is drawn through the tip of the 14.66-kN vector. Likewise a line representing the direction of the pin reaction $\mathbf{A}$, determined from the concurrency established with the free-body diagram, is drawn through the tail of the 14.66-kN vector. The intersection of the lines representing vectors $\mathbf{T}$ and $\mathbf{A}$ establishes the magnitudes T and A necessary to make the vector sum of the forces equal to zero. These magnitudes are scaled from the diagram. The x- and y-components of $\mathbf{A}$ may be constructed on the force polygon if desired.

Helpful Hints

① The justification for this step is Varignon's theorem, explained in Art. 2/4. Be prepared to take full advantage of this principle frequently.

② The calculation of moments in two-dimensional problems is generally handled more simply by scalar algebra than by the vector cross product $\mathbf{r} \times \mathbf{F}$. In three dimensions, as we will see later, the reverse is often the case.

③ The direction of the force at A could be easily calculated if desired. However, in designing the pin A or in checking its strength, it is only the magnitude of the force that matters.



Graphical solution