3rd Lecture

Solution of differential equations

11 To solve a differential equation, we have to find the function for which the equation is true. This means that we have to manipulate the equation so as to eliminate all the derivatives and leave a relationship between y and x. The rest of this particular Programme is devoted to the various methods of solving *first-order differential equations*. Second-order equations will be dealt with in the next Programme.

So, for the first method, move on to Frame 12

12 Method 1: *By direct integration*

If the equation can be arranged in the form $\frac{dy}{dx} = f(x)$, then the equation can be solved by simple integration.

Example 1

$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

Then $y = \int (3x^2 - 6x + 5)dx = x^3 - 3x^2 + 5x + C$

i.e. $y = x^3 - 3x^2 + 5x + C$

As always, of course, the constant of integration must be included. Here it provides the one arbitrary constant which we always get when solving a first-order differential equation.

Method 2: *By separating the variables*

If the given equation is of the form $\frac{dy}{dx} = f(x, y)$, the variable y on the right-hand side prevents solving by direct integration. We therefore have to devise some other method of solution.

Let us consider equations of the form $\frac{dy}{dx} = f(x).F(y)$ and of the form $\frac{dy}{dx} = \frac{f(x)}{F(y)}$, i.e. equations in which the right-hand side can be expressed as products or quotients of functions of x or of y.

A few examples will show how we proceed.

Example 1

Solve $\frac{dy}{dx} = \frac{2x}{y+1}$

We can rewrite this as $(y+1)\frac{dy}{dx} = 2x$

Now integrate both sides with respect to x:

$$\int (y+1)\frac{dy}{dx}\,dx = \int 2x\,dx \quad \text{i.e.}$$

$$\int (y+1)\,dy = \int 2x\,dx$$

and this gives $\frac{y^2}{2} + y = x^2 + C$

Example 2

Solve $\quad \dfrac{dy}{dx} = (1+x)(1+y)$

$$\frac{1}{1+y}\frac{dy}{dx} = 1 + x$$

Integrate both sides with respect to x:

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int (1+x)\,dx \qquad \therefore \quad \int \frac{1}{1+y}\,dy = \int (1+x)\,dx$$

$$\ln(1+y) = x + \frac{x^2}{2} + C$$

Example 3

Solve $\quad \dfrac{dy}{dx} = \dfrac{1+y}{2+x} \qquad (1)$

This can be written as $\dfrac{1}{1+y}\dfrac{dy}{dx} = \dfrac{1}{2+x}$

Integrate both sides with respect to x:

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int \frac{1}{2+x}\,dx$$

$$\therefore \quad \int \frac{1}{1+y}\,dy = \int \frac{1}{2+x}\,dx \qquad (2)$$

$$\therefore \quad \ln(1+y) = \ln(2+x) + C$$

It is convenient to write the constant C as the logarithm of some other constant A:

$$\ln(1+y) = \ln(2+x) + \ln A = \ln A(2+x)$$

$$\therefore \quad 1 + y = A(2+x)$$

Example 4

Solve $\quad \dfrac{dy}{dx} = \dfrac{y^2 + xy^2}{x^2y - x^2}$

First express the RHS in 'x-factors' and 'y-factors':

$$\frac{dy}{dx} = \frac{y^2(1+x)}{x^2(y-1)}$$

Now rearrange the equation so that we have the 'y-factors' and dy on the LHS and the 'x-factors' and dx on the RHS.

So we get ....

$$\boxed{\frac{y-1}{y^2}\,dy = \frac{1+x}{x^2}\,dx}$$

We now add the integral signs:

$$\int \frac{y-1}{y^2}\,dy = \int \frac{1+x}{x^2}\,dx$$

and complete the solution:

$$\int \left\{\frac{1}{y} - y^{-2}\right\} dy = \int \left\{x^{-2} + \frac{1}{x}\right\} dx$$

$$\therefore \quad \ln y + y^{-1} = \ln x - x^{-1} + C$$

$$\therefore \quad \ln y + \frac{1}{y} = \ln x - \frac{1}{x} + C$$

Example 5

Solve $$\frac{dy}{dx} = \frac{y^2 - 1}{x}$$

Rearranging, we have $$dy = \frac{y^2 - 1}{x} dx$$

$$\frac{1}{y^2 - 1} dy = \frac{1}{x} dx$$

$$\int \frac{1}{y^2 - 1} dy = \int \frac{1}{x} dx$$

$$= \int \frac{1}{(y-1)(y+1)} dy$$

Which gives …………

$$\boxed{\frac{1}{2} \ln \frac{y-1}{y+1} = \ln x + C}$$

$$\therefore \quad \ln\frac{y-1}{y+1} = 2\ln x + \ln A$$

$$\therefore \quad \frac{y-1}{y+1} = Ax^2$$

$$y - 1 = Ax^2(y+1)$$

Example 2

Solve $x\frac{dy}{dx} = 5x^3 + 4$

In this case, $\frac{dy}{dx} = 5x^2 + \frac{4}{x}$ So, $\int dy = \int \left(5x^2 + \frac{4}{x}\right) dx$

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$$y = \frac{5x^3}{3} + 4\ln x + C$$

Example 3

Find the particular solution of the equation $e^x\frac{dy}{dx} = 4$, given that $y = 3$ when $x = 0$.

First rewrite the equation in the form $\frac{dy}{dx} = \frac{4}{e^x} = 4e^{-x}$.

Then $y = \int 4e^{-x}\,dx = -4e^{-x} + C$

Knowing that when $x = 0$, $y = 3$, we can evaluate C in this case, so that the required particular solution is $y =$ . . . .

$$y = -4e^{-x} + 7$$

Example 6

Solve $xy\dfrac{dy}{dx} = \dfrac{x^2+1}{y+1}$

First of all, rearrange the equation into the form:

$$F(y)dy = f(x)dx$$

i.e. arrange the 'y-factors' and dy on the LHS and the 'x-factors' and dx on the RHS.

What do you get?

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$$y(y+1)dy = \frac{x^2+1}{x}dx$$

$$xy\frac{dy}{dx} = \frac{x^2+1}{y+1} \qquad \therefore\ xy\,dy = \frac{x^2+1}{y+1}\,dx \qquad \therefore\ y(y+1)\,dy = \frac{x^2+1}{x}\,dx$$

So we now have $\displaystyle\int (y^2+y)\,dy = \int\left(x+\frac{1}{x}\right)dx$

$$\frac{y^3}{3} + \frac{y^2}{2} = \frac{x^2}{2} + \ln x + C$$