8th Lecture

3 *Complex roots* to the auxiliary equation

Now let us see what we get when the roots of the auxiliary equation are complex.

Suppose $m = \alpha \pm j\beta$, i.e. $m_1 = \alpha + j\beta$ and $m_2 = \alpha - j\beta$. Then the solution would be of the form:

$$y = Ce^{(\alpha+j\beta)x} + De^{(\alpha-j\beta)x} = Ce^{\alpha x}.e^{j\beta x} + De^{\alpha x}.e^{-j\beta x}$$

$$= e^{\alpha x}\{Ce^{j\beta x} + De^{-j\beta x}\}$$

Now from our previous work on complex numbers, we know that:

$$e^{jx} = \cos x + j\sin x$$

$$e^{-jx} = \cos x - j\sin x$$

$$\text{and that} \begin{cases} e^{j\beta x} = \cos\beta x + j\sin\beta x \\ e^{-j\beta x} = \cos\beta x - j\sin\beta x \end{cases}$$

Solve $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+9y=0$

Auxiliary equation: $m^2+4m+9=0$

$$\therefore\ m=\frac{-4\pm\sqrt{16-36}}{2}=\frac{-4\pm\sqrt{-20}}{2}=\frac{-4\pm 2j\sqrt{5}}{2}=-2\pm j\sqrt{5}$$

In this case $\alpha=-2$ and $\beta=\sqrt{5}$

Solution is: $y=e^{-2x}\left(A\cos\sqrt{5}x+B\sin\sqrt{5}x\right)$

Now you can solve this one: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0$$

Auxiliary equation: $m^2 - 2m + 10 = 0$

$$m = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = 1 \pm j3$$

$$y = e^x(A\cos 3x + B\sin 3x)$$

Equations of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$

Auxiliary equation: $am^2 + bm + c = 0$

1 *Roots real and different* $m = m_1$ and $m = m_2$

Solution is $y = Ae^{m_1x} + Be^{m_2x}$

2 *Real and equal roots* $m = m_1$ (twice)

Solution is $y = e^{m_1x}(A + Bx)$

3 *Complex roots* $m = \alpha \pm j\beta$

Solution is $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

Here are some examples:

Example 1

$$\frac{d^2y}{dx^2} + 16y = 0 \quad \therefore \; m^2 = -16 \quad \therefore \; m = \pm j4$$

$$\therefore \; y = A\cos 4x + B\sin 4x$$

Example 2

$$\frac{d^2y}{dx^2} - 3y = 0 \quad \therefore \; m^2 = 3 \quad \therefore \; m = \pm\sqrt{3}$$

$$y = A\cosh\sqrt{3}x + B\sinh\sqrt{3}x$$

Similarly

Example 3

$$\frac{d^2y}{dx^2} + 5y = 0$$

$$\boxed{y = A\cos\sqrt{5}x + B\sin\sqrt{5}x}$$

Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 2\sin 4x$

(1) *CF* This will be the same as in the previous example, since the LHS of this equation is the same

i.e. $y = Ae^{2x} + Be^{3x}$

(2) *PI* The general form of the PI in this case will be ....

Note: Although the RHS is $f(x) = 2\sin 4x$, it is necessary to include the full general function $y = C\cos 4x + D\sin 4x$ since, in finding the derivatives, the cosine term will also give rise to $\sin 4x$.

So we have:

$$y = C\cos 4x + D\sin 4x$$

$$\frac{dy}{dx} = -4C\sin 4x + 4D\cos 4x$$

$$\frac{d^2y}{dx^2} = -16C\cos 4x - 16D\sin 4x$$

$$-16C\cos 4x - 16D\sin 4x + 20C\sin 4x - 20D\cos 4x$$
$$+ 6C\cos 4x + 6D\sin 4x = 2\sin 4x$$

$$(20C - 10D)\sin 4x - (10C + 20D)\cos 4x = 2\sin 4x$$

$$\begin{aligned} 20C - 10D &= 2 \qquad & 40C - 20D &= 4 \\ 10C + 20D &= 0 \qquad & 10C + 20D &= 0 \end{aligned} \Bigg\} 50C = 4 \qquad \therefore\ C = \frac{2}{25}$$

$$D = -\frac{1}{25}$$

In each case the PI is $y = \frac{1}{25}(2\cos 4x - \sin 4x)$

The CF was $y = Ae^{2x} + Be^{3x}$

The general solution is: $y = Ae^{2x} + Be^{3x} + \frac{1}{25}(2\cos 4x - \sin 4x)$

Example 3

Solve $\dfrac{d^2y}{dx^2} + 14\dfrac{dy}{dx} + 49y = 4e^{5x}$

First we have to find the CF. To do this we solve the equation . . . .

$$\boxed{\frac{d^2y}{dx^2} + 14\frac{dy}{dx} + 49y = 0}$$

Correct. So start off by writing down the auxiliary equation, which is . . . .

$$\boxed{m^2 + 14m + 49 = 0}$$

This gives $(m+7)(m+7) = 0$, i.e. $m = -7$ (twice)

$\therefore$ The CF is $y = e^{-7x}(A + Bx)$ (1)

Now for the PI. To find this we take the general form of the RHS of the given equation, i.e. we assume $y =$ . . . .

$$\boxed{y = Ce^{5x}}$$

Right. So we now differentiate twice which gives us:

$$\frac{dy}{dx} = \text{. . . . .} \quad \text{and} \quad \frac{d^2y}{dx^2} = \text{. . . . .}$$

$$\boxed{\frac{dy}{dx} = 5Ce^{5x}, \quad \frac{d^2y}{dx^2} = 25\,Ce^{5x}}$$

The equation now becomes:

$$25Ce^{5x} + 14.5Ce^{5x} + 49Ce^{5x} = 4e^{5x}$$

Dividing through by e^{5x}: $25C + 70C + 49C = 4$

$$144C = 4 \quad \therefore \; C = \frac{1}{36}$$

The PI is $y = \dfrac{e^{5x}}{36}$

So there we are. The CF is $y = e^{-7x}(A + Bx)$

and the PI is $y = \dfrac{e^{5x}}{36}$

and the complete general solution is therefore …………

$$\boxed{y = e^{-7x}(A + Bx) + \frac{e^{5x}}{36}}$$

Example 4

Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 2\sin 2x$

(1) *To find CF* Solve LHS $= 0$ $\therefore m^2 + 6m + 10 = 0$

$$\therefore m = \frac{-6 \pm \sqrt{36 - 40}}{2} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm j$$

$$y = e^{-3x}(A\cos x + B\sin x) \qquad (1)$$

(2) *To find PI* Assume the general form of the RHS

i.e. $\boxed{y = C\cos 2x + D\sin 2x}$

$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 2\sin 2x$ and equate coefficients of $\sin 2x$ and of $\cos 2x$.

Off you go then. Find the PI on your own.

$$y = C\cos 2x + D\sin 2x$$

$$\therefore \frac{dy}{dx} = -2C\sin 2x + 2D\cos 2x$$

$$\therefore \frac{d^2y}{dx^2} = -4C\cos 2x - 4D\sin 2x$$

Substituting in the equation gives:

$$-4C\cos 2x - 4D\sin 2x - 12C\sin 2x + 12D\cos 2x$$
$$+ 10C\cos 2x + 10D\sin 2x = 2\sin 2x$$

$$(6C + 12D)\cos 2x + (6D - 12C)\sin 2x = 2\sin 2x$$

$$6C + 12D = 0 \quad \therefore \; C = -2D$$

$$6D - 12C = 2 \quad \therefore \; 6D + 24D = 2 \quad \therefore \; 30D = 2 \quad \therefore \; D = \frac{1}{15}$$

$$\therefore \; C = -\frac{2}{15}$$

$$\text{PI is } y = \frac{1}{15}(\sin 2x - 2\cos 2x)$$

So the CF is $y = e^{-3x}(A\cos x + B\sin x)$

and the PI is $y = \dfrac{1}{15}(\sin 2x - 2\cos 2x)$

The complete general solution is therefore

$$\boxed{y = e^{-3x}(A\cos x + B\sin x) + \frac{1}{15}(\sin 2x - 2\cos 2x)}$$