2^{nd} Lecture

Formation of differential equations

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Differential equations may be formed in practice from a consideration of the physical problems to which they refer. Mathematically, they can occur when arbitrary constants are eliminated from a given function. Here are a few examples.

Example 1

Consider $y = A\sin x + B\cos x$, where A and B are two arbitrary constants.

If we differentiate, we get:

$$\frac{dy}{dx} = A\cos x - B\sin x$$

and $$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$$

which is identical to the original equation, but with the sign changed.

i.e. $$\frac{d^2y}{dx^2} = -y \qquad \therefore \quad \frac{d^2y}{dx^2} + y = 0$$

This is 2nd order differential equation

Example 2

Form a differential equation from the function $y = x + \frac{A}{x}$.

We have $y = x + \frac{A}{x} = x + Ax^{-1}$

$$\therefore \quad \frac{dy}{dx} = 1 - Ax^{-2} = 1 - \frac{A}{x^2}$$

From the given equation, $\frac{A}{x} = y - x \quad \therefore \quad A = x(y - x)$

$$\therefore \quad \frac{dy}{dx} = 1 - \frac{x(y-x)}{x^2}$$

$$= 1 - \frac{y-x}{x} = \frac{x-y+x}{x} = \frac{2x-y}{x}$$

$$\therefore \quad x\frac{dy}{dx} = 2x - y$$

This is first order deferential equation

Example 3

Form the differential equation for $y = Ax^2 + Bx$.

We have $$y = Ax^2 + Bx \quad (1)$$

$$\therefore \frac{dy}{dx} = 2Ax + B \quad (2)$$

$$\therefore \frac{d^2y}{dx^2} = 2A \quad (3) \qquad A = \frac{1}{2}\frac{d^2y}{dx^2}$$

Substitute for $2A$ in (2): $\frac{dy}{dx} = x\frac{d^2y}{dx^2} + B$

$$\therefore B = \frac{dy}{dx} - x\frac{d^2y}{dx^2}$$

Substituting for A and B in (1), we have:

$$y = x^2 . \frac{1}{2}\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)$$

$$= \frac{x^2}{2} . \frac{d^2y}{dx^2} + x . \frac{dy}{dx} - x^2 . \frac{d^2y}{dx^2}$$

$$\therefore y = x\frac{dy}{dx} - \frac{x^2}{2} . \frac{d^2y}{dx^2}$$

This is 2nd order differential equation

If we collect our last few results together, we have:

$y = A\sin x + B\cos x$ gives the equation $\dfrac{d^2y}{dx^2} + y = 0$ (2nd order)

$y = Ax^2 + Bx$ gives the equation $y = x\dfrac{dy}{dx} - \dfrac{x^2}{2}\cdot\dfrac{d^2y}{dx^2}$ (2nd order)

$y = x + \dfrac{A}{x}$ gives the equation $x\dfrac{dy}{dx} = 2x - y$ (1st order)

If we were to investigate the following, we should also find that:

$y = Axe^x$ gives the differential equation $x\dfrac{dy}{dx} - y(1+x) = 0$ (1st order)

$y = Ae^{-4x} + Be^{-6x}$ gives the differential equation $\dfrac{d^2y}{dx^2} + 10\dfrac{dy}{dx} + 24y = 0$ (2nd order)

Some of the functions give 1st-order equations: some give 2nd-order equations. Now look at the five results above and see if you can find any distinguishing features in the functions which decide whether we obtain a 1st-order equation or a 2nd-order equation in any particular case.

Home Work

Correct, and in the same way:

A function with 3 arbitrary constants would give a 3rd order equation.

So, without working each out in detail, we can say that:

(a) $y = e^{-2x}(A + Bx)$ would give a differential equation of ………… order.

(b) $y = A\dfrac{x-1}{x+1}$ would give a differential equation of ………… order.

(c) $y = e^{3x}(A\cos 3x + B\sin 3x)$ would give a differential equation of ………… order.

Because

(a) and (c) each have 2 arbitrary constants,

while (b) has only 1 arbitrary constant.

Similarly:

(a) $x^2 \frac{dy}{dx} + y = 1$ is derived from a function having .... arbitrary constants.

(b) $\cos^2 x \frac{dy}{dx} = 1 - y$ is derived from a function having .... arbitrary constants.

(c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = e^{2x}$ is derived from a function having .... arbitrary constants.

Solution of differential equations

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To solve a differential equation, we have to find the function for which the equation is true. This means that we have to manipulate the equation so as to eliminate all the derivatives and leave a relationship between y and x. The rest of this particular Programme is devoted to the various methods of solving *first-order differential equations*. Second-order equations will be dealt with in the next Programme.

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Method 1: *By direct integration*

If the equation can be arranged in the form $\frac{dy}{dx} = f(x)$, then the equation can be solved by simple integration.

Example 1

$$\frac{dy}{dx} = 3x^2 - 6x + 5$$

Then $y = \int (3x^2 - 6x + 5)dx = x^3 - 3x^2 + 5x + C$

i.e. $y = x^3 - 3x^2 + 5x + C$

As always, of course, the constant of integration must be included. Here it provides the one arbitrary constant which we always get when solving a first-order differential equation.

Example 2

$$\text{Solve} \quad x\frac{dy}{dx} = 5x^3 + 4$$

$$\text{In this case,} \quad \frac{dy}{dx} = 5x^2 + \frac{4}{x} \qquad \text{So,} \quad y = \ldots\ldots\ldots\ldots$$

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$$y = \frac{5x^3}{3} + 4\ln x + C$$

As you already know from your work on integration, the value of C cannot be determined unless further information about the function is given. In this present form, the function is called the *general solution* (or *primitive*) of the given equation.

If we are told the value of y for a given value of x, C can be evaluated and the result is then a *particular solution* of the equation.

Example 3

Find the particular solution of the equation $e^x \dfrac{dy}{dx} = 4$, given that $y = 3$ when $x = 0$.

First rewrite the equation in the form $\dfrac{dy}{dx} = \dfrac{4}{e^x} = 4e^{-x}$.

Then $y = \int 4e^{-x}\,dx = -4e^{-x} + C$

Knowing that when $x = 0$, $y = 3$, we can evaluate C in this case, so that the required particular solution is $y = \ldots\ldots\ldots\ldots$

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$y = -4e^{-x} + 7$

Method 2: *By separating the variables*

If the given equation is of the form $\frac{dy}{dx} = f(x, y)$, the variable y on the right-hand side prevents solving by direct integration. We therefore have to devise some other method of solution.

Let us consider equations of the form $\frac{dy}{dx} = f(x).F(y)$ and of the form $\frac{dy}{dx} = \frac{f(x)}{F(y)}$, i.e. equations in which the right-hand side can be expressed as products or quotients of functions of x or of y.

A few examples will show how we proceed.

Example 1

Solve $\frac{dy}{dx} = \frac{2x}{y+1}$

We can rewrite this as $(y+1)\frac{dy}{dx} = 2x$

Now integrate both sides with respect to x:

$$\int (y+1)\frac{dy}{dx}\,dx = \int 2x\,dx \quad \text{i.e.}$$

$$\int (y+1)\,dy = \int 2x\,dx$$

and this gives $\frac{y^2}{2} + y = x^2 + C$

Example 2

Solve $\dfrac{dy}{dx} = (1+x)(1+y)$

$$\frac{1}{1+y}\frac{dy}{dx} = 1+x$$

Integrate both sides with respect to x:

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int (1+x)\,dx \qquad \therefore \int \frac{1}{1+y}\,dy = \int (1+x)\,dx$$

$$\ln(1+y) = x + \frac{x^2}{2} + C$$

Example 3

Solve $\dfrac{dy}{dx} = \dfrac{1+y}{2+x}$ (1)

This can be written as $\dfrac{1}{1+y}\dfrac{dy}{dx} = \dfrac{1}{2+x}$

Integrate both sides with respect to x:

$$\int \frac{1}{1+y}\frac{dy}{dx}\,dx = \int \frac{1}{2+x}\,dx$$

$$\therefore \int \frac{1}{1+y}\,dy = \int \frac{1}{2+x}\,dx \qquad (2)$$

$$\therefore \ln(1+y) = \ln(2+x) + C$$

It is convenient to write the constant C as the logarithm of some other constant A:

$$\ln(1+y) = \ln(2+x) + \ln A = \ln A(2+x)$$

$$\therefore 1+y = A(2+x)$$