# 6th Lecture

## Bernoulli's equation

These are equations of the form:

$$\frac{dy}{dx} + Py = Qy^n$$

where, as before, $P$ and $Q$ are functions of $x$ (or constants).

The trick is the same every time:

(a) Divide both sides by $y^n$. This gives:

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$$

(b) Now put $z = y^{1-n}$

so that, differentiating, $\frac{dz}{dx} = \ldots\ldots\ldots\ldots$

$$\boxed{\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}}$$

So we have:

$$\frac{dy}{dx} + Py = Qy^n \qquad (1)$$

$$\therefore\ y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \qquad (2)$$

Put $z = y^{1-n}$ so that $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

If we now multiply (2) by $(1-n)$ we shall convert the first term into $\frac{dz}{dx}$.

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)Py^{1-n} = (1-n)Q$$

Remembering that $z = y^{1-n}$ and that $\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$, this last line can now be written $\frac{dz}{dx} + P_1 z = Q_1$ with $P_1$ and $Q_1$ functions of $x$.

**Example 1**

Solve $\frac{dy}{dx} + \frac{1}{x}y = xy^2$

(a) Divide through by $y^2$, giving ............

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}.y^{-1} = x$$

(b) Now put $z = y^{1-n}$, i.e. in this case $z = y^{1-2} = y^{-1}$

$$z = y^{-1} \quad \therefore \quad \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$$

(c) Multiply through the equation by $(-1)$, to make the first term $\frac{dz}{dx}$.

$$-y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = -x$$

so that $\frac{dz}{dx} - \frac{1}{x}z = -x$ which is of the form $\frac{dz}{dx} + Pz = Q$ so that you can now

Check the working:

$$\frac{dz}{dx} - \frac{1}{x}z = -x$$

$$IF = e^{\int P dx} \qquad \int P\, dx = \int -\frac{1}{x}\, dx = -\ln x$$

$$\therefore \; IF = e^{-\ln x} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$

$$z.IF = \int Q.IF\, dx \quad \therefore \; z\frac{1}{x} = \int -x.\frac{1}{x}\, dx$$

$$\therefore \; \frac{z}{x} = \int -1\, dx = -x + C$$

$$\therefore \; z = Cx - x^2$$

But $z = y^{-1} \quad \therefore \; \frac{1}{y} = Cx - x^2 \quad \therefore \; y = (Cx - x^2)^{-1}$

## Example 2

Solve $2y - 3\frac{dy}{dx} = y^4 e^{3x}$

$$2y - 3\frac{dy}{dx} = y^4 e^{3x}$$

$$\therefore \quad \frac{dy}{dx} - \frac{2}{3}y = -\frac{y^4 e^{3x}}{3}$$

$$\therefore \quad y^{-4}\frac{dy}{dx} - \frac{2}{3}y^{-3} = -\frac{e^{3x}}{3}$$

Put $z = y^{1-4} = y^{-3}$ $\quad \therefore \quad \frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$

Multiplying through by $(-3)$, the equation becomes:

$$-3y^{-4}\frac{dy}{dx} + 2y^{-3} = e^{3x}$$

$$\text{i.e.} \quad \frac{dz}{dx} + 2z = e^{3x}$$

$$\text{IF} = e^{\int P dx} \qquad \int P\,dx = \int 2\,dx = 2x \quad \therefore \quad \text{IF} = e^{2x}$$

$$\therefore \quad ze^{2x} = \int e^{3x}e^{2x}dx = \int e^{5x}dx$$

$$= \frac{e^{5x}}{5} + C$$

But $z = y^{-3}$ $\quad \therefore \quad \frac{e^{2x}}{y^3} = \frac{e^{5x} + A}{5}$

$$\therefore \quad y^3 = \frac{5e^{2x}}{e^{5x} + A}$$

# Second-order differential equations

## Learning outcomes

*When you have completed this Programme you will be able to:*

- Use the auxiliary equation to solve certain second-order homogeneous equations
- Use the complementary function and the particular integral to solve certain second-order inhomogeneous equations

Many practical problems in engineering give rise to second-order differential equations of the form

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

where $a$, $b$ and $c$ are constant coefficients and $f(x)$ is a given function of $x$. By the end of this Programme you will have no difficulty with equations of this type.

Let us first take the case where $f(x)=0$, so that the equation becomes

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

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The solution, then, of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ is seen to be

$$y=A\,e^{m_1x}+B\,e^{m_2x}$$

where $A$ and $B$ are two arbitrary constants and $m_1$ and $m_2$ are the roots of the quadratic equation $am^2+bm+c=0$.

**Example 1**

Solve $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 24$

(1) *CF* Solve LHS = 0 $\therefore\ m^2 - 5m + 6 = 0$

$$\therefore\ (m-2)(m-3) = 0 \quad \therefore\ m = 2 \text{ and } m = 3$$

$$\therefore\ y = Ae^{2x} + Be^{3x} \qquad (1)$$

(2) *PI* $f(x) = 24$, i.e. a constant. Assume $y = C$

$$\text{Then } \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} = 0$$

Substituting in the given equation:

$0 - 5(0) + 6C = 24 \qquad C = 24/6 = 4$

$\therefore$ PI is $y = 4$ (2)

General solution is $y = \text{CF} + \text{PI}$, i.e. $y = \underbrace{Ae^{2x} + Be^{3x}}_{\text{CF}}\underbrace{+4}_{\text{PI}}$

**Example**

For the equation $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$, the auxiliary equation is $2m^2 + 5m + 6 = 0$.

In the same way, for the equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, the auxiliary equation is ............

$$m^2 + 3m + 2 = 0$$

Since the auxiliary equation is always a quadratic equation, the values of $m$ can be determined in the usual way.

i.e. if $m^2 + 3m + 2 = 0$

$$(m+1)(m+2) = 0 \quad \therefore \; m = -1 \text{ and } m = -2$$

$\therefore$ the solution of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ is

$$y = Ae^{-x} + Be^{-2x}$$

In the same way, if the auxiliary equation were $m^2 + 4m - 5 = 0$, this factorizes into $(m+5)(m-1) = 0$ giving $m = 1$ or $-5$, and in this case the solution would be ............

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$$y = Ae^{x} + Be^{-5x}$$

## 1 *Real and different roots* to the auxiliary equation

### Example 1

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Auxiliary equation: $m^2 + 5m + 6 = 0$

$\therefore \ (m+2)(m+3) = 0 \quad \therefore \ m = -2$ or $m = -3$

$\therefore$ Solution is $y = Ae^{-2x} + Be^{-3x}$

### Example 2

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$$

Auxiliary equation: $m^2 - 7m + 12 = 0$

$(m-3)(m-4) = 0 \quad \therefore \ m = 3$ or $m = 4$

So the solution is . . . . . . . . . . . .

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$$y = Ae^{3x} + Be^{4x}$$

**Example 1**

Solve $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = 0$

Auxiliary equation: $m^2 + 4m + 4 = 0$

$(m+2)(m+2) = 0 \quad \therefore \; m = -2$ (twice)

The solution is: $y = e^{-2x}(A + Bx)$

Here is another:

**Example 2**

Solve $\dfrac{d^2y}{dx^2} + 10\dfrac{dy}{dx} + 25y = 0$

Auxiliary equation: $m^2 + 10m + 25 = 0$

$(m+5)^2 = 0 \quad \therefore \; m = -5$ (twice)

$y = e^{-5x}(A + Bx)$