9TH lecture

Introduction to Laplace transforms

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the Laplace transform of an expression by using the integral definition
- Obtain inverse Laplace transforms with the help of a Table of Laplace transforms
- Derive the Laplace transform of the derivative of an expression
- Solve first-order, constant-coefficient, inhomogeneous differential equations using the Laplace transform
- Derive further Laplace transforms from known transforms
- Use the Laplace transform to obtain the solution to linear, constant-coefficient, inhomogeneous differential equations of second and higher order

The Laplace transform

All the differential equations you have looked at so far have had solutions containing a number of unknown integration constants A, B, C etc. The values of these constants have then been found by applying boundary conditions to the solution, a procedure that can often prove to be tedious. Fortunately, for a certain type of differential equation there is a method of obtaining the solution where these unknown integration constants are evaluated *during the process of solution*. Furthermore, rather than employing integration as the way of unravelling the differential equation, you use straightforward algebra.

The method hinges on what is called the *Laplace transform*. If $f(x)$ represents some expression in x defined for $x \geq 0$, the *Laplace transform* of $f(x)$, denoted by $L\{f(x)\}$, is defined to be:

$$L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx} f(x)\,\mathrm{d}x$$

where s is a variable whose values are chosen so as to ensure that the semi-infinite integral converges. More will be said about the variable s in Frame 3.

For now, what would you say is the Laplace transform $f(x) = 2$ for $x \geq 0$?

Substitute for $f(x)$ in the integral above and then perform the integration.
The answer is in the next frame

$$L\{2\} = \frac{2}{s} \text{ provided } s > 0$$

Because:

$$L\{f(x)\} = \int_{x=0}^{\infty} e^{-sx} f(x)\,\mathrm{d}x$$

so

$$\begin{aligned} L\{2\} &= \int_{x=0}^{\infty} e^{-sx} 2\,\mathrm{d}x \\ &= 2\left[\frac{e^{-sx}}{-s}\right]_{x=0}^{\infty} \\ &= 2(0 - (-1/s)) \\ &= \frac{2}{s} \end{aligned}$$

Notice that $s > 0$ is demanded because if $s < 0$ then $e^{-sx} \to \infty$ as $x \to \infty$ and if $s = 0$ then $L\{2\}$ is not defined (in both of these two cases the integral diverges), so that

$$L\{2\} = \frac{2}{s} \text{ provided } s > 0$$

By the same reasoning, if k is some constant then

$$L\{k\} = \frac{k}{s} \text{ provided } s > 0$$

Now, how about the Laplace transform of $f(x) = e^{-kx}$, $x \geq 0$ where k is a constant?

Go back to the integral definition and work it out.
Again, the answer is in the next frame

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$$\boxed{L\{e^{-kx}\} = \frac{1}{s+k} \text{ provided } s > -k}$$

Because

$$\begin{aligned}
L\{e^{-kx}\} &= \int_{x=0}^{\infty} e^{-sx}e^{-kx}\,\mathrm{d}x \\
&= \int_{x=0}^{\infty} e^{-(s+k)x}\,\mathrm{d}x \\
&= \left[\frac{e^{-(s+k)x}}{-(s+k)}\right]_{x=0}^{\infty} \\
&= \left(0 - \left(-\frac{1}{(s+k)}\right)\right) \quad s+k>0 \text{ is demanded to ensure that the integral converges at both limits} \\
&= \frac{1}{(s+k)} \quad \text{provided } s+k>0\text{, that is provided } s > -k
\end{aligned}$$

Examples

1 Find the Laplace transform of each of the following. In each case $f(x)$ is defined for $x \geq 0$:

(a) $f(x) = -3$ (b) $f(x) = e$ (c) $f(x) = e^{2x}$

(d) $f(x) = -5e^{-3x}$ (e) $f(x) = 2e^{7x-2}$

1 (a) $f(x) = -3$

Because $L\{k\} = \frac{k}{s}$ provided $s > 0$, $L\{-3\} = -\frac{3}{s}$ provided $s > 0$

(b) $f(x) = e$

Because $L\{k\} = \frac{k}{s}$ provided $s > 0$, $L\{e\} = \frac{e}{s}$ provided $s > 0$

(c) $f(x) = e^{2x}$

Because $L\{e^{-kx}\} = \frac{1}{s+k}$ provided $s > -k$, $L\{e^{2x}\} = \frac{1}{s-2}$ provided $s > 2$

(d) $f(x) = -5e^{-3x}$

$$L\{-5e^{-3x}\} = \int_{x=0}^{\infty} e^{-sx}(-5e^{-3x})\,dx = -5\int_{x=0}^{\infty} e^{-sx}e^{-3x}\,dx = -5L\{e^{-3x}\}$$

$L\{-5e^{-3x}\} = -\frac{5}{s+3}$ provided $s > -3$

(e) $f(x) = 2e^{7x-2}$

$$L\{2e^{7x-2}\} = \int_{x=0}^{\infty} e^{-sx}(2e^{7x-2})\,dx = 2e^{-2}\int_{x=0}^{\infty} e^{-sx}e^{7x}\,dx = 2e^{-2}L\{e^{7x}\}$$

$$L\{2e^{7x-2}\} = \frac{2e^{-2}}{s-7} \text{ provided } s > 7$$

The inverse Laplace transform

The Laplace transform is an expression in the variable s which is denoted by $F(s)$. It is said that $f(x)$ and $F(s) = L\{f(x)\}$ form a *transform pair*. This means that if $F(s)$ is the *Laplace transform* of $f(x)$ then $f(x)$ is the *inverse Laplace transform* of $F(s)$. We write:

$$f(x) = L^{-1}\{F(s)\}$$

There is no simple integral definition of the inverse transform so you have to find it by working backwards. For example:

$$\text{if } f(x) = 4 \text{ then the Laplace transform } L\{f(x)\} = F(s) = \frac{4}{s}$$

so

$$\text{if } F(s) = \frac{4}{s} \text{ then the inverse Laplace transform } L^{-1}\{F(s)\} = f(x) = 4$$

It is this ability to find the Laplace transform of an expression and then reverse it that makes the Laplace transform so useful in the solution of differential equations, as you will soon see.

For now, what is the inverse Laplace transform of $F(s) = \dfrac{1}{s-1}$?

$$L^{-1}\{F(s)\} = f(x) = e^x$$

Because you know that:

$L\{e^{-kx}\} = \frac{1}{s+k}$ you can say that $L^{-1}\left\{\frac{1}{s+k}\right\} = e^{-kx}$

so when $k = -1$, $L^{-1}\left\{\frac{1}{s-1}\right\} = e^{-(-1)x} = e^x$

Table of Laplace transforms

$f(x) = L^{-1}\{F(s)\}$	$F(s) = L\{f(x)\}$	
k	$\frac{k}{s}$	$s > 0$
e^{-kx}	$\frac{1}{s+k}$	$s > -k$

Examples

2 Find the inverse Laplace transform of each of the following:

(a) $F(s) = -\frac{1}{s}$ (b) $F(s) = \frac{1}{s-5}$ (c) $F(s) = \frac{3}{s+2}$

(d) $F(s) = -\frac{3}{4s}$ (e) $F(s) = \frac{1}{2s-3}$

2 (a) $F(s) = -\frac{1}{s}$

Because $L^{-1}\left\{\frac{k}{s}\right\} = k$, $L^{-1}\left\{-\frac{1}{s}\right\} = L^{-1}\left\{\frac{-1}{s}\right\} = -1$

(b) $F(s) = \frac{1}{s-5}$

Because $L^{-1}\left\{\frac{1}{s+k}\right\} = e^{-kx}$, $L^{-1}\left\{\frac{1}{s-5}\right\} = e^{-(-5)x} = e^{5x}$

(c) $F(s) = \frac{3}{s+2}$

Because $L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2x}$ and $L\{3e^{-2x}\} = 3L\{e^{-2x}\} = \frac{3}{s+2}$ so

$L^{-1}\left\{\frac{3}{s+2}\right\} = 3e^{-2x}$

(d) $F(s) = -\dfrac{3}{4s}$

$$F(s) = -\frac{3}{4s} = \frac{(-3/4)}{s} \text{ so that } L^{-1}\left\{-\frac{3}{4s}\right\} = L^{-1}\left\{\frac{-3/4}{s}\right\} = -3/4$$

(e) $F(s) = \dfrac{1}{2s-3}$

$$F(s) = \frac{1}{2s-3} = \frac{\frac{1}{2}}{s-\frac{3}{2}} \text{ so that } f(x) = L^{-1}\left\{\frac{1}{2s-3}\right\} = L^{-1}\left\{\frac{\frac{1}{2}}{s-\frac{3}{2}}\right\} = \frac{1}{2}e^{\frac{3}{2}x}$$

Laplace transform of a derivative

Before you can use the Laplace transform to solve a differential equation you need to know the Laplace transform of a derivative. Given some expression $f(x)$ with Laplace transform $L\{f(x)\} = F(s)$, the Laplace transform of the derivative $f'(x)$ is:

$$L\{f'(x)\} = \int_{x=0}^{\infty} e^{-sx} f'(x)\,\mathrm{d}x$$

This can be integrated by parts as follows:

$$\begin{aligned} L\{f'(x)\} &= \int_{x=0}^{\infty} e^{-sx} f'(x)\,\mathrm{d}x \\ &= \int_{x=0}^{\infty} u(x)\mathrm{d}v(x) \\ &= \Big[u(x)v(x)\Big]_{x=0}^{\infty} - \int_{x=0}^{\infty} v(x)\mathrm{d}u(x) \end{aligned}$$

(the Parts formula – see Programme 15, Frame 21)

where $u(x) = e^{-sx}$ so $\mathrm{d}u(x) = -se^{-sx}\mathrm{d}x$ and where $\mathrm{d}v(x) = f'(x)\mathrm{d}x$ so $v(x) = f(x)$.

Therefore, substitution in the Parts formula gives:

$$L\{f'(x)\} = \left[e^{-sx}f(x)\right]_{x=0}^{\infty} + s\int_{x=0}^{\infty} e^{-sx}f(x)\mathrm{d}x$$

$$= (0 - f(0)) + sF(s) \text{ assuming } e^{-sx}f(x) \to 0 \text{ as } x \to \infty$$

That is:

$$L\{f'(x)\} = sF(s) - f(0)$$

So the Laplace transform of the derivative of $f(x)$ is given in terms of the Laplace transform of $f(x)$ itself and the value of $f(x)$ when $x = 0$. Before you use this fact just consider two properties of the Laplace transform in the next frame.

Two properties of Laplace transforms

Both the Laplace transform and its inverse are *linear transforms*, by which is meant that:

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is:*

$$L\{f(x) \pm g(x)\} = L\{f(x)\} \pm L\{g(x)\}$$

and $$L^{-1}\{F(s) \pm G(s)\} = L^{-1}\{F(s)\} \pm L^{-1}\{G(s)\}$$

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is:*

$L\{kf(x)\} = kL\{f(x)\}$ and $L^{-1}\{kF(s)\} = kL^{-1}\{F(s)\}$ where k is a constant

Armed with this information let's try a simple differential equation. By using

$$L\{f'(x)\} = sF(s) - f(0)$$

take the Laplace transform of both sides of the equation

$$f'(x) + f(x) = 1 \text{ where } f(0) = 0$$

and find an expression for the Laplace transform $F(s)$.

$$F(s) = \frac{1}{s(s+1)}$$

Because, taking Laplace transforms of both sides of the equation you have that:

$L\{f'(x) + f(x)\} = L\{1\}$ The Laplace transform of the left-hand side equals the Laplace transform of the right-hand side

That is:

$L\{f'(x)\} + L\{f(x)\} = L\{1\}$ The transform of a sum is the sum of the transforms.

From what you know about the Laplace transform of $f(x)$ and its derivative $f'(x)$ this gives:

$$[sF(s) - f(0)] + F(s) = \frac{1}{s}$$

That is:

$(s+1)F(s) - f(0) = \frac{1}{s}$ and you are given that $f(0) = 0$ so

$(s+1)F(s) = \frac{1}{s}$, that is $F(s) = \frac{1}{s(s+1)}$

Well done. Now, separate the right-hand side into partial fractions.

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

Because

Assume that $\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$ then, $1 = A(s+1) + Bs$ from which you find that $A = 1$ and $B = -1$ so that $F(s) = \frac{1}{s} - \frac{1}{s+1}$

That was straightforward enough. Now take the inverse Laplace transform and find the solution to the differential equation.

$$f(x) = 1 - e^{-x}$$

Because

$$f(x) = L^{-1}\{F(s)\}$$

$$= L^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$ The inverse Laplace transform of a difference is the difference of the inverse transforms

$$= 1 - e^{-x}$$ Using the Table of Laplace transforms

Table of Laplace transforms

$f(x) = L^{-1}\{F(s)\}$	$F(s) = L\{f(x)\}$	
k	$\frac{k}{s}$	$s > 0$
e^{-kx}	$\frac{1}{s+k}$	$s > -k$
xe^{-kx}	$\frac{1}{(s+k)^2}$	$s > -k$

Revision exercise

Solve each of the following differential equations:

(a) $f'(x) - f(x) = 2$ where $f(0) = 0$

(b) $f'(x) + f(x) = e^{-x}$ where $f(0) = 0$

(c) $f'(x) + f(x) = 3$ where $f(0) = -2$

(d) $f'(x) - f(x) = e^{2x}$ where $f(0) = 1$

(e) $3f'(x) - 2f(x) = 4e^{-x} + 2$ where $f(0) = 0$

(a) $f'(x) - f(x) = 2$ where $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$$sF(s) - f(0) - F(s) = \frac{2}{s} \text{ so that } F(s) = \frac{2}{s(s-1)} = -\frac{2}{s} + \frac{2}{s-1}$$

The inverse transform then gives the solution as

$$f(x) = -2 + 2e^x = 2(e^x - 1)$$

(b) $f'(x) + f(x) = e^{-x}$ where $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$$sF(s) - f(0) + F(s) = \frac{1}{s+1} \text{ so that } F(s) = \frac{1}{(s+1)^2}$$

The Table of inverse transforms then gives the solution as $f(x) = xe^{-x}$

(c) $f'(x) + f(x) = 3$ where $f(0) = -2$

Taking Laplace transforms of both sides of this equation gives:

$$sF(s) - f(0) + F(s) = \frac{3}{s} \text{ so that}$$

$$F(s) = -\frac{2}{s+1} + \frac{3}{s(s+1)} = \frac{3-2s}{s(s+1)} = \frac{3}{s} - \frac{5}{s+1}$$

The inverse transform then gives the solution as $f(x) = 3 - 5e^{-x}$

· (d) $f'(x) - f(x) = e^{2x}$ where $f(0) = 1$

Taking Laplace transforms of both sides of this equation gives:

$$sF(s) - f(0) - F(s) = \frac{1}{s-2} \text{ giving } (s-1)F(s) - 1 = \frac{1}{s-2}$$

$$\text{so that } F(s) = \frac{1}{s-1} + \frac{1}{(s-1)(s-2)} = \frac{1}{s-2}$$

The inverse transform then gives the solution as $f(x) = e^{2x}$

(e) $3f'(x) - 2f(x) = 4e^{-x} + 2$ where $f(0) = 0$

Taking Laplace transforms of both sides of this equation gives:

$$3[sF(s) - f(0)] - 2F(s) = \frac{4}{s+1} + \frac{2}{s} = \frac{6s+2}{s(s+1)} \text{ so that}$$

$$F(s) = \frac{6s+2}{s(s+1)(3s-2)} = \frac{27}{5}\left(\frac{1}{3s-2}\right) - \frac{1}{s} - \frac{4}{5}\left(\frac{1}{s+1}\right)$$

$$= \frac{27}{15}\left(\frac{1}{s-\frac{2}{3}}\right) - \frac{1}{s} - \frac{4}{5}\left(\frac{1}{s+1}\right)$$

The inverse transform then gives the solution as:

$$f(x) = \frac{9}{5}e^{2x/3} - \frac{4}{5}e^{-x} - 1$$