4^{th} Lecture

Method 3: *Homogeneous equations – by substituting y= vx*

Here is an equation:

$$\frac{dy}{dx} = \frac{x+3y}{2x}$$

This looks simple enough, but we find that we cannot express the RHS in the form of 'x-factors' and 'y-factors', so we cannot solve by the method of separating the variables.

In this case we make the substitution $y = vx$, where v is a function of x. So $y = vx$. Differentiate with respect to x (using the product rule):

$$\therefore \quad \frac{dy}{dx} = v.1 + x\frac{dv}{dx} = v + x\frac{dv}{dx}$$

Also $\quad \dfrac{x+3y}{2x} = \dfrac{x+3vx}{2x} = \dfrac{1+3v}{2}$

The equation now becomes $v + x\dfrac{dv}{dx} = \dfrac{1+3v}{2}$

$$\therefore \quad x\frac{dv}{dx} = \frac{1+3v}{2} - v$$

$$= \frac{1+3v-2v}{2} = \frac{1+v}{2}$$

$$\therefore \quad x\frac{dv}{dx} = \frac{1+v}{2}$$

The given equation is now expressed in terms of v and x, and in this form we find that we can solve by separating the variables. Here goes:

$$\int \frac{2}{1+v} \mathrm{d}v = \int \frac{1}{x} \mathrm{d}x$$

$$\therefore \; 2\ln(1+v) = \ln x + C = \ln x + \ln A$$

$$(1+v)^2 = Ax$$

But $y = vx \quad \therefore \; v = \left\{\frac{y}{x}\right\} \quad \therefore \; \left(1 + \frac{y}{x}\right)^2 = Ax$

which gives $(x+y)^2 = Ax^3$

Solve $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{xy}$

Here, all terms of the RHS are of degree 2, i.e. the equation is homogeneous.

$\therefore$ We substitute $y = vx$ (where v is a function of x)

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

and $$\frac{x^2 + y^2}{xy} = \frac{x^2 + v^2x^2}{vx^2} = \frac{1 + v^2}{v}$$

The equation now becomes:

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1 + v^2}{v} - v$$

$$= \frac{1 + v^2 - v^2}{v} = \frac{1}{v}$$

$$\therefore x\frac{dv}{dx} = \frac{1}{v}$$

Now you can separate the variables and get the result in terms of v and x.

$$\boxed{\frac{v^2}{2} = \ln x + C}$$

Because

$$\int v \, dv = \int \frac{1}{x} \, dx$$

$$\therefore \quad \frac{v^2}{2} = \ln x + C$$

All that remains now is to express v back in terms of x and y. The substitution we used was $y = vx \quad \therefore \; v = \dfrac{y}{x}$

$$\therefore \quad \frac{1}{2}\left(\frac{y}{x}\right)^2 = \ln x + C$$

$$y^2 = 2x^2(\ln x + C)$$

Example 2

Solve $\dfrac{dy}{dx} = \dfrac{2xy + 3y^2}{x^2 + 2xy}$

Is this a homogeneous equation? If you think so, what are your reasons?

$y = vx$, where v is a function of x

Right. That is the key to the whole process.

$$\frac{dy}{dx} = \frac{2xy + 3y^2}{x^2 + 2xy}$$

So express each side of the equation in terms of v and x:

$$\frac{dy}{dx} = \dots\dots\dots\dots$$

and $$\frac{2xy + 3y^2}{x^2 + 2xy} = \dots\dots\dots\dots$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$
$$\frac{2xy + 3y^2}{x^2 + 2xy} = \frac{2vx^2 + 3v^2x^2}{x^2 + 2vx^2} = \frac{2v + 3v^2}{1 + 2v}$$

So that $v + x\dfrac{dv}{dx} = \dfrac{2v + 3v^2}{1 + 2v}$

$$x\frac{dv}{dx} = \frac{2v+3v^2}{1+2v} - v$$

$$= \frac{2v+3v^2-v-2v^2}{1+2v}$$

$$x\frac{dv}{dx} = \frac{v+v^2}{1+2v}$$

$$\int \frac{1+2v}{v+v^2}\,dv = \int \frac{1}{x}\,dx$$

Integrating both sides, we can now obtain the solution in terms of v and x. What do you get?

$$\ln(v+v^2) = \ln x + C = \ln x + \ln A$$

$$\therefore\ v+v^2 = Ax$$

We have almost finished the solution. All that remains is to express v back in terms of x and y.

Remember the substitution was $y = vx$, so that $v = \dfrac{y}{x}$

So finish it off.

Then move on

$$xy + y^2 = Ax^3$$

Because

$$v + v^2 = Ax \text{ and } v = \frac{y}{x} \qquad \therefore \; \frac{y}{x} + \frac{y^2}{x^2} = Ax$$

$$xy + y^2 = Ax^3$$

Example 3

Solve $(x^2 + y^2)\dfrac{dy}{dx} = xy$

Here is the solution in full.

$$(x^2 + y^2)\frac{dy}{dx} = xy \qquad \therefore \quad \frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

Put $y = vx \quad \therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

and $\dfrac{xy}{x^2 + y^2} = \dfrac{vx^2}{x^2 + v^2x^2} = \dfrac{v}{1 + v^2}$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{v}{1 + v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1 + v^2} - v$$

$$x\frac{dv}{dx} = \frac{v - v - v^3}{1 + v^2} = \frac{-v^3}{1 + v^2}$$

$$\therefore \quad \int \frac{1+v^2}{v^3}\,\mathrm{d}v = -\int \frac{1}{x}\,\mathrm{d}x$$

$$\therefore \quad \int \left(v^{-3} + \frac{1}{v}\right)\mathrm{d}v = -\ln x + C$$

$$\therefore \quad \frac{-v^{-2}}{2} + \ln v = -\ln x + \ln A$$

$$\ln v + \ln x + \ln K = \frac{1}{2v^2} \qquad (\ln K = -\ln A)$$

$$\ln Kvx = \frac{1}{2v^2}$$

But $\quad v = \dfrac{y}{x} \qquad \therefore \quad \ln Ky = \dfrac{x^2}{2y^2}$

$$2y^2 \ln Ky = x^2$$

1 $(x-y)\dfrac{dy}{dx}=x+y \quad \therefore \quad \dfrac{dy}{dx}=\dfrac{x+y}{x-y}$

Put $\quad y=vx \quad \therefore \quad \dfrac{dy}{dx}=v+x\dfrac{dv}{dx} \qquad \dfrac{x+y}{x-y}=\dfrac{1+v}{1-v}$

$$\therefore \quad v+x\frac{dv}{dx}=\frac{1+v}{1-v} \quad \therefore \quad x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v-v+v^2}{1-v}=\frac{1+v^2}{1-v}$$

$$\therefore \quad \int\frac{1-v}{1+v^2}dv=\int\frac{1}{x}dx \quad \therefore \quad \int\left\{\frac{1}{1+v^2}-\frac{v}{1+v^2}\right\}dv=\ln x+C$$

$$\therefore \quad \tan^{-1}v-\frac{1}{2}\ln(1+v^2)=\ln x+\ln A$$

But $\quad v=\dfrac{y}{x} \quad \therefore \quad \tan^{-1}\left\{\dfrac{y}{x}\right\}=\ln A+\ln x+\dfrac{1}{2}\ln\left(1+\dfrac{y^2}{x^2}\right)$

Equation **2** gives the solution: $\boxed{\dfrac{2x}{x-y}=\ln x+C}$

Method 4: ***Linear equations – use of integrating factor***

The equation we have just solved is an example of a set of equations of the form $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x (or constants). This equation is called a *linear equation of the first order* and to solve any such equation, we multiply both sides by an *integrating factor* which is always

So: *To solve a differential equation of the form*

$$\frac{dy}{dx} + Py = Q$$

where P and Q are constants or functions of x, multiply both sides by the integrating factor $e^{\int P dx}$

Example 1

To solve $\dfrac{dy}{dx} - y = x$

If we compare this with $\dfrac{dy}{dx} + Py = Q$, we see that in this case

$$P = -1 \text{ and } Q = x.$$

The integrating factor is always $e^{\int P\,dx}$ and here $P = -1$.

$\therefore \int P\,dx = -x$ and the integrating factor is therefore ....

$$\boxed{e^{-x}}$$

We therefore multiply both sides by e^{-x}.

$$\therefore\ e^{-x}\frac{dy}{dx} - ye^{-x} = xe^{-x}$$

$$\frac{d}{dx}\left\{e^{-x}y\right\} = xe^{-x} \quad \therefore\ ye^{-x} = \int xe^{-x}\,dx$$

The RHS integral can now be determined by integrating by parts:

$$ye^{-x} = x(-e^{-x}) + \int e^{-x}\,\mathrm{d}x = -xe^{-x} - e^{-x} + C$$

$$\therefore\ y = -x - 1 + Ce^{x} \quad \therefore\ y = Ce^{x} - x - 1$$

Example 2

Solve $x\dfrac{dy}{dx}+y=x^3$

First we divide through by x to reduce the first term to a single $\dfrac{dy}{dx}$

i.e. $\dfrac{dy}{dx}+\dfrac{1}{x}.y=x^2$

Compare with $\left[\dfrac{dy}{dx}+Py=Q\right]$ $\therefore$ $P=\dfrac{1}{x}$ and $Q=x^2$

$\text{IF}=e^{\int P dx}$ $\int P\,dx=\int\dfrac{1}{x}\,dx=\ln x$

$\therefore$ $\text{IF}=e^{\ln x}=x$ $\therefore$ $\text{IF}=x$

The solution is $y.\text{IF}=\int Q.\text{IF}\,dx$

so $yx=\int x^2.x\,dx=\int x^3\,dx=\dfrac{x^4}{4}+C$ $\therefore$ $xy=\dfrac{x^4}{4}+C$

Example 3

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} + y\cot x = \cos x$

Compare with $\left[\frac{\mathrm{d}y}{\mathrm{d}x} + Py = Q\right] \quad \therefore \begin{cases} P = \cot x \\ Q = \cos x \end{cases}$

$\text{IF} = e^{\int P\mathrm{d}x} \qquad \int P\,\mathrm{d}x = \int \cot x\,\mathrm{d}x = \int \frac{\cos x}{\sin x}\,\mathrm{d}x = \ln \sin x$

$\therefore \text{IF} = e^{\ln \sin x} = \sin x$

$y.\text{IF} = \int Q.\text{IF}\,\mathrm{d}x \quad \therefore y\sin x = \int \sin x \cos x\,\mathrm{d}x = \frac{\sin^2 x}{2} + C$

$\therefore y = \frac{\sin x}{2} + C\,\text{cosec}\,x$

Now here is another.

Example 4

Solve $(x+1)\frac{dy}{dx} + y = (x+1)^2$

The first thing is to ....

Divide through by $(x+1)$

Correct, since we must reduce the coefficient of $\frac{dy}{dx}$ to 1.

$$\therefore \quad \frac{dy}{dx} + \frac{1}{x+1}.y = x+1$$

Compare with $\frac{dy}{dx} + Py = Q$

In this case $P = \frac{1}{x+1}$ and $Q = x+1$

$$\boxed{\text{IF} = x + 1}$$

Because

$$\int P\,\mathrm{d}x = \int \frac{1}{x+1}\,\mathrm{d}x = \ln(x+1)$$

$$\therefore\ \text{IF} = e^{\ln(x+1)} = (x+1)$$

The solution is always $y.\text{IF} = \int Q.\text{IF}\,\mathrm{d}x$

and we know that, in this case, $\text{IF} = x + 1$ and $Q = x + 1$.

So finish off the solution and then move on to Frame 56

$$\boxed{y = \frac{(x+1)^2}{3} + \frac{C}{x+1}}$$

Example 5

Solve $x\dfrac{dy}{dx} - 5y = x^7$

In this case, $P =$ .... $Q =$ ....

$$P = -\frac{5}{x} \quad Q = x^6$$

Because if

$$x\frac{dy}{dx} - 5y = x^7$$

$$\therefore \quad \frac{dy}{dx} - \frac{5}{x}.y = x^6$$

Compare with $\left[\dfrac{dy}{dx} + Py = Q\right] \quad \therefore \; P = -\dfrac{5}{x}; \quad Q = x^6$

So integrating factor, IF = ...

$$\boxed{\mathrm{IF} = x^{-5} = \frac{1}{x^5}}$$

Because

$$\mathrm{IF} = e^{\int P\mathrm{d}x} \qquad \int P\,\mathrm{d}x = -\int \frac{5}{x}\,\mathrm{d}x = -5\ln x$$

$$\therefore\ \mathrm{IF} = e^{-5\ln x} = e^{\ln(x^{-5})} = x^{-5} = \frac{1}{x^5}$$

So the solution is:

$$y.\frac{1}{x^5} = \int x^6.\frac{1}{x^5}\,\mathrm{d}x$$

$$\frac{y}{x^5} = \int x\,\mathrm{d}x = \frac{x^2}{2} + C \qquad \therefore\ y = \ldots\ldots\ldots\ldots$$

$$\boxed{y = \frac{x^7}{2} + Cx^5}$$

Example 6

Solve $(1 - x^2)\dfrac{dy}{dx} - xy = 1.$

$$\boxed{y\sqrt{1-x^2} = \sin^{-1} x + C}$$

Here is the working in detail. Follow it through.

$$(1 - x^2)\frac{dy}{dx} - xy = 1$$

$$\therefore \quad \frac{dy}{dx} - \frac{x}{1-x^2}.y = \frac{1}{1-x^2}$$

$$\text{IF} = e^{\int P dx} \qquad \int P\,dx = \int \frac{-x}{1-x^2}\,dx = \frac{1}{2}\ln(1-x^2)$$

$$\therefore \quad \text{IF} = e^{\frac{1}{2}\ln(1-x^2)} = e^{\ln\{(1-x^2)^{\frac{1}{2}}\}} = (1-x^2)^{\frac{1}{2}}$$

$$\text{Now } y.\text{IF} = \int Q.\text{IF}\,dx$$

$$\therefore \quad y\sqrt{1-x^2} = \int \frac{1}{1-x^2}\sqrt{1-x^2}.dx$$

$$= \int \frac{1}{\sqrt{1 - x^2}} \, dx = \sin^{-1} x + C$$

$$y\sqrt{1 - x^2} = \sin^{-1} x + C$$