Biomedical Engineering
Engineering Analysis
Dr. Shahad Ali Hammood
shahad.ali@mustaqbal-college.edu.iq

Lecture (8)

Matrices

1. Linearly Dependent & Linearly Independent Vectors

The vectors $V_1, V_2, V_3, \ldots, V_m$ are linearly dependent if and only if $|V_1 \ \ V_2 \ \ V_3 \ \ \ldots \ \ V_m| = 0$. If $|V_1 \ \ V_2 \ \ V_3 \ \ \ldots \ \ V_m| \neq 0$ then $V_1, V_2, V_3, \ldots, V_m$ are linearly independent.

Example 1

$V_1 = (3,6,-1), V_2 = (8,2,-4), V_3 = (1,-1,1)$

Since $\begin{vmatrix} 3 & 8 & 1 \\ 6 & 2 & -1 \\ -1 & -4 & 1 \end{vmatrix} = 3\begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} - 8\begin{vmatrix} 6 & -1 \\ -1 & 1 \end{vmatrix} + 1\begin{vmatrix} 6 & 2 \\ -1 & -4 \end{vmatrix}$

$3(2-4) - 8(6-1) + (-24+2) = -6 - 40 - 22 = -68 \neq 0$

Then V_1, V_2, V_3 are linearly independent

$V_1 = (2,4,6), V_2(1,3,3), V_3 = (1,2,3)$

Since $\begin{vmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 6 & 3 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ 3 & 3 \end{vmatrix} - 1\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} + 1\begin{vmatrix} 4 & 3 \\ 6 & 3 \end{vmatrix}$

$= 2(9-6) - (12-12) + (12-18) = 6 - 0 - 6 = 0$

Then V_1, V_2, V_3 are linearly dependent

2. Linearly Dependent & Linearly Independent Functions

If $y_1, y_2, y_3, \ldots, y_m$ are functions of x then $y_1, y_2, y_3, \ldots, y_m$ are linearly dependent if $w(y_1, y_2, y_3, \ldots, y_m) = 0$ where

$$w(y_1, y_2, y_3, \ldots\ldots y_m) = \begin{vmatrix} y_1 & y_2 \ldots. & y_m \\ y'_1 & y'_2 \ldots. & y'_m \\ y_1^{(m-1)}\vdots & y_2^{(m-1)} \ldots & y_m^{(m-1)} \end{vmatrix}$$

If $w(y_1, y_2, y_3, \ldots, y_m) \neq 0$ then $y_1, y_2, y_3, \ldots, y_m$ are linearly independent

Example: 1) $y_1 = e^x, y_2 = x^2$

$$w(y_1, y_2) = \begin{vmatrix} e^x & x^2 \\ e^x & 2x \end{vmatrix} = 2xe^x - x^2e^x = xe^x(2 - x) \neq 0$$

So y_1 and y_2 are linearly independent

2) $y_1 = 4e^x, y_2 = 2\,e^x$

$$w(y_1, y_2) = \begin{vmatrix} 4e^x & 2e^x \\ 4e^x & 2e^x \end{vmatrix} = 8e^{2x} - 8e^{2x} = 0$$ So y_1 and y_2 are linearly dependent

Gauss Elimination Method

The system of linear equations is denoted by

$AX = B$ *where A is a matrix, X and B are vectors*

This system is solved by using the Gauss elimination method as shown below.

Example:

The system

$$x_1 - 2x_2 = 3$$

$$4x_1 + 6x_2 = 5$$

Can be written as

$$\begin{bmatrix} 1 & -2 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

So, by using the Gauss elimination method as follows:

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 4 & 6 & 5 \end{array}\right]$$

$$4R_1 - R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & -14 & 7 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -2 \\ 0 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} \rightarrow x_1 - 2x_2 = 3\,, -14x_2 = 7$$

$$-14x_2 = 7 \rightarrow x_2 = -\frac{1}{2} \rightarrow x_1 = 2\left(-\frac{1}{2}\right) + 3 = 2$$

The solution is $\begin{bmatrix} 2 \\ -1/2 \end{bmatrix}$

Matrix Inverse

We can use the Gauss elimination method to find the inverse of a matrix, we know that a matrix A and its inverse A^{-1} must satisfy the equation $A \times A^{-1} = I$ (identity matrix). The following example illustrates the procedure.

Example: Find A^{-1} if

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

Solution

$$\left[\begin{matrix} 2 & 1 \\ 1 & 4 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right] \to \frac{1}{2}R_1 \to \left[\begin{matrix} 1 & 1/2 \\ 1 & 4 \end{matrix}\middle|\begin{matrix} 1/2 & 0 \\ 0 & 1 \end{matrix}\right]$$

$$R_2 - R_1 \to \left[\begin{matrix} 1 & 1/2 \\ 0 & 7/2 \end{matrix}\middle|\begin{matrix} 1/2 & 0 \\ -1/2 & 1 \end{matrix}\right] \to \frac{2}{7}R_2 \to \left[\begin{matrix} 1 & 1/2 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 1/2 & 0 \\ -1/7 & 2/7 \end{matrix}\right]$$

$$R_1 - \frac{1}{2}R_2 \to \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{matrix}\right] \to A^{-1} = \begin{bmatrix} 4/7 & -1/7 \\ -1/7 & 2/7 \end{bmatrix}$$

Note: The matrix A has an inverse if $|A| \neq 0$

Eigen Values & Eigen Vectors

Let A be an $n \times n$ matrix, a real or complex number λ is called an Eigen value of A if det(A- λI)= 0 and with [A- λI]x= 0 then x is called an Eigen vector with respect to λ where I is the identity matrix.

Example: Find the Eigen values and Eigen vectors of A if

a) A= $\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, b) A= $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, $c)\ A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

Sol.: a) To find Eigen values, we have $\det(A - \lambda I) = 0$ then

$$A - \lambda I = \begin{vmatrix} 1-\lambda & 2 \\ -1 & -2-\lambda \end{vmatrix} = 0 \to (1-\lambda)(-2-\lambda) + 2 = 0 \to -2 - \lambda + 2\lambda + \lambda^2 + 2 = 0$$

$$\to \lambda^2 + \lambda = 0 \to \lambda(\lambda + 1) = 0 \to \lambda_1 = 0,\ \lambda_2 = -1$$

To find the Eigen vectors, we have $[A - \lambda I]X = 0$

For $\lambda_1 = 0 \to \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_1 + 2x_2 = 0 \to x_1 = -2x_2$

$Let\ x_2 = 1 \to x_1 = -2 \to The\ Eigen\ vector\ for\ \lambda_1 = 0\ is\ \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $\lambda_2 = -1 \to \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$2x_1 + 2x_2 = 0 \rightarrow 2x_1 = -2x_2 \rightarrow x_1 = -x_2$

$Let\ x_2 = 1 \rightarrow x_1 = -1 \rightarrow The\ Eigen\ vector\ for\ \lambda_2 = -1\ is \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

b) To find Eigen values, we have det(A-λI)= 0 then

$$A - \lambda I = \begin{vmatrix} 1-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda)^2 - 6 = 0 \rightarrow 1 - 2\lambda + \lambda^2 - 6 = 0$$

$$\rightarrow \lambda^2 - 2\lambda - 5 = 0 \rightarrow \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{+2 \pm \sqrt{4+20}}{2} = 1 \pm \sqrt{6}$$

$\lambda_1 = 1 + \sqrt{6},\ \lambda_2 = 1 - \sqrt{6}$

$To\ find\ the\ Eigen\ vectors, we\ have\ [A - \lambda I]X = 0$

For $\lambda_1 = 1 + \sqrt{6} \rightarrow \begin{bmatrix} -\sqrt{6} & 3 \\ 2 & -\sqrt{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$-\sqrt{6}\,x_1 + 3x_2 = 0 \rightarrow -\sqrt{6}\,x_1 = -3x_2 \rightarrow x_1 = \frac{3}{\sqrt{6}} x_2$

$Let\ x_2 = 1 \rightarrow x_1 = \frac{3}{\sqrt{6}} \rightarrow The\ Eigen\ vector\ for\ \lambda_1 = 1 + \sqrt{6}\ is \begin{bmatrix} \frac{3}{\sqrt{6}} \\ 1 \end{bmatrix}$

For $\lambda_2 = 1 - \sqrt{6} \rightarrow \begin{bmatrix} \sqrt{6} & 3 \\ 2 & \sqrt{6} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\sqrt{6}\,x_1 + 3x_2 = 0 \rightarrow \sqrt{6}\,x_1 = -3x_2 \rightarrow x_1 = \frac{-3}{\sqrt{6}} x_2$

$Let\ x_2 = 1 \rightarrow x_1 = \frac{-3}{\sqrt{6}} \rightarrow The\ Eigen\ vector\ for\ \lambda_2 = 1 - \sqrt{6}\ is \begin{bmatrix} \frac{-3}{\sqrt{6}} \\ 1 \end{bmatrix}$

c) To find Eigen values, we have det(A-λI)= 0 then

$$A - \lambda I = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \rightarrow (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 1 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\rightarrow -(1-\lambda)^2(1+\lambda) = 0 \rightarrow -(1 - 2\lambda + \lambda^2)(1+\lambda) \rightarrow -\big((\lambda-1)(\lambda-1)\big)(1+\lambda) = 0$$

$-\lambda_1 + 1 = 0 \rightarrow \lambda_1 = 1, \quad -\lambda_2 + 1 = 0 \rightarrow \lambda_2 = 1, \qquad -1 - \lambda_3 = 0 \rightarrow \lambda_3 = -1$

$\rightarrow \lambda_1 = 1, \quad \lambda_2 = 1, \quad and\ \lambda_3 = -1$

$To\ find\ the\ Eigen\ vectors, we\ have\ [\ A - \lambda I]X = 0$

For $\lambda_{1,2} = 1 \rightarrow \begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = 0 \quad , \quad -x_2 = 0,\ x_3 = 0, -2x_3 = 0$

The Eigen vector for $\lambda_{1,2} = 1\ is \begin{vmatrix} a \\ 0 \\ 0 \end{vmatrix}$

Each choice of a gives us an Eigen vector associated with λ=1

$$For\ \lambda_3 = -1 \rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$2x_1 - x_2 = 0 \ \rightarrow 2x_1 = x_2 \ \rightarrow 2x_2 + x_3 = 0 \ \rightarrow 2x_2 = -x_3$

$Let\ x_1 = 1 \ \rightarrow x_2 = 2 \ \rightarrow x_3 = -4$

The Eigen vector for $\lambda_3 = -1\ is \ \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

ملاحظات عن الكسور الجزئية

(Partial Fraction)

الكسور الجزئية هي عملية عكسية لتوحيد المقامات وشروطها

١. درجة البسط اقل من درجة المقام او تساويها
٢. توزيع المقام
٣. نفرض قيمة لل x مناسبة لتصفير احد المتغيرات لايجاد المتغير الاخر او الحل بطريقة المعادلات.

حل مثال ١٣ في محاضرة ٦ بالتفصيل

Example 13. Solve the equation $\frac{d^2x}{dt^2} - 3\ \frac{dx}{dt} + 2x = 2\ e^{3t} \quad given\ that\ t = 0, x = 5,\ \frac{dx}{dt} = 7$

$Sol:. (s^2\ x' - sx_\circ - \ x_1) - 3\ (s\ x' - x_\circ) + 2\ x' = \dfrac{2}{s-3}$

$s^2\ x' - 5s - 7 - 3sx' + 15 + 2\ x' = \dfrac{2}{s-3}$

$x'(\ s^2 - 3s + 2) - 5s + 8 = \dfrac{2}{s-3}$

$x'\ (s-1)(s-2) = \dfrac{2}{s-3} + 5s - 8$

$x'\ (s-1)(s-2) = \dfrac{2 + 5s\ (s-3) - 8\ (s-3)}{s-3}$

$x' = \dfrac{5\ s^2 - 23s + 26}{(s-3)(s-1)(s-2)}$

$\dfrac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)} = \dfrac{A}{s-1} + \dfrac{B}{s-2} + \dfrac{C}{s-3}$

$5\ s^2 - 23s + 26 = A\ (s-2)(s-3) + B(s-1)(s-3) + c(s-1)(s-2)$
$5\ s^2 - 23s + 26 = A(s^2 - 3s - 2s + 6) + B(\ s^2 - 3s - s + 3) + C(s^2 - 2s - s + 2)$
$5s^2 - 23s + 26 = A\ s^2 - 5As + 6A + Bs^2 - 4Bs + 3B + Cs^2 - 3Cs + 2C$
$5s^2 = A\ s^2 + B\ s^2 + C\ s^2 \rightarrow A + B + C = 5 \qquad \dots\dots\dots 1$
$-23s = \ -5As - 4Bs - 3Cs$
$-23 = -5A - 4B - 3C \qquad \dots\dots\dots 2$

$26 = 6A + 3B + 2C$ ………… 3

$(A + B + C = 5) * -2$

$6A + 3B + 2C = 26$

$-2A - 2B - 2C = -10$

$6A + 3B + 2C = 26$

$4A + B = 16 \rightarrow 4A = 16 - B \rightarrow A = \frac{16 - B}{4}$ ……… . 1

$A + B + C = 5 \rightarrow \frac{16 - B}{4} + B + C = 5 \rightarrow \frac{16 - B + 4B}{4} + C = 5 \rightarrow \frac{16 + 3B}{4} + C = 5$

$4 + \frac{3}{4} B + C = 5 \rightarrow C = 1 - \frac{3}{4} B$……. 2

$-23 = -5A - 4B - 3C \rightarrow -23 = -5\left(\frac{16 - B}{4}\right) - 4B - 3(1 - \frac{3}{4}B)$

$-23 = \frac{-5 \times 16 + 5B}{4} - 4B - 3 + \frac{9}{4}B$

$-23 + 3 = -20 + \frac{5}{4}B - 4B + \frac{9}{4}B \rightarrow -20 = -20 + \frac{14}{4}B - 4B \rightarrow 0 = \frac{-2}{4}B \rightarrow$

$B = 0 \; sub \; in \; 1\&2$

A=4, B= 0, C=1

$x' = \frac{4}{s - 1} + 0 + \frac{1}{s - 3}$

$x = 4\, e^{t} + e^{3t}$