Biomedical Engineering
Engineering Analysis
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Lecture (6)

Laplace Transformation Theorem

1) First shift theory

$if\ f(t) \rightarrow f(s)$

$e^{at}\ f(t) \rightarrow F(s-a)$

Example 1. Find L.T.

$$f(t) = \frac{\sqrt{t}}{e^{2t}}$$

Sol.

$$f(t) = e^{-2t}\ (t)^{\frac{1}{2}} \rightarrow\ t^{\frac{1}{2}} = \frac{\Gamma 3/2}{s^{\frac{3}{2}}}$$

$$e^{-2t}\ t^{1/2} = \frac{\Gamma\ 3/2}{(s+2)^{3/2}}$$

Example 2. Find L.T for

$f(t) =\ e^{t}\ sin^{2}t$

Sol.

$Sin^2t = 1/2 - 1/2\ cos2t$

$$f(t) = \frac{1}{2} - \frac{1}{2}\ cos2t$$

$$F(s) = \frac{\frac{1}{2}}{s} - \frac{1}{2}\frac{s}{s^2+4}$$

$$e^{t}.sin^{2}t = \frac{1/2}{(s-1)} - \frac{1}{2}\frac{s-1}{(s-1)^2+4}$$

2) Derivative of Transformation

$if\ f(t) \rightarrow f(s)$

$$t^n\, f(t) \rightarrow -\frac{d}{ds}\, f(s) = +\frac{d^2}{d^2 s}\, f(s) = -\frac{d^3}{d^3 s}\, f(s)$$

Example 3. Find the solve of $t.sin3t$

Sol.

$$f(t) = sin3t \rightarrow \frac{a}{s^2+a^2} = \frac{3}{s^2+9}$$

$$f(t) = -\frac{d}{ds}\, f(s) \rightarrow -\frac{d}{ds}\left(\frac{3}{s^2+9)}\right) = -\frac{(s^2+9).0 - 3\times 2s}{(s^2+9)^2} = +\frac{6s}{(s^2+9)^2}$$

Example 4. Find the solve of $t^2.e^{-3t}$

Sol.

$$f(t) = t^2 \rightarrow \frac{n!}{s^{n+1}} \rightarrow \frac{2}{s^3} \rightarrow \frac{2}{(s-a)^3} = \frac{2}{(s+3)^3} \qquad (First\ Shifting)$$

$$f(t) = e^{-3t} \rightarrow \frac{1}{s+3}$$

$$t^2.e^{-3t} = +\frac{d^2}{ds^2}\,\mathrm{f(s)} \rightarrow \frac{d^2}{ds^2}\left(\frac{1}{s+3}\right) \rightarrow \frac{(s+3)^2.0+2\times(s+3)}{(s+3)^4} = \frac{2}{(s+3)^3} \qquad (\,Derivative\ of\ transformation)$$

3) Integral of Transform

$$\frac{f(t)}{t^n} = \int_s^\infty F(s)ds$$

Example 5. Find the L.T. $f(t) = \frac{\sin(t)}{t}$

Sol.

$$f(t) = sint \rightarrow \frac{1}{s^2+1} \rightarrow \int_s^\infty \frac{1}{s^2+1}\, ds = [tan^{-1}(s)]_s^\infty \rightarrow tan^{-1}(\infty) - tan^{-1}(s) = f(s) = \frac{\pi}{2} - tan^{-1}(s)$$

Example 6. Find the solve of $f(t) = \frac{\mathrm{e}^{-6t} - \mathrm{e}^{4t}}{\mathrm{t}}$

Sol.

$$f(t) = e^{-6t} - e^{4t} \rightarrow F(s) = \frac{1}{s+6} - \frac{1}{s-4}$$

$$F(s) = \int_s^\infty \left(\frac{1}{s+6}\right) - \left(\frac{1}{s-4}\right) ds \rightarrow [\ln(s+6) - \ln(s-4)]_s^\infty$$

$$= [\ln(\infty+6) - \ln(\infty-4)] - [\ln(s+6) - \ln(s-4)]$$

4) Transform of Integral

$$\int_0^t f(t)dt \rightarrow \frac{F(s)}{s}$$

Example 7. $f(t) = \int_0^t e^{-t}\sin(t)$

$$f(t) = sint \rightarrow F(s) = \frac{1}{s^2+1} \rightarrow \frac{1}{(s-a)^2+1} = \frac{1}{(s+1)^2+1} \quad first\ shift\ theory$$

$\int_0^t e^{-t}\sin(t) \rightarrow \frac{1/(s+1)^2+1}{s} = \frac{1}{s\,(s+1)^2+1}$ transform of integral

Inverse Laplace Transformation

$$F(t) \leftrightarrow F(s) \begin{cases} Table \\ Laplace\ Transformation\ Theorem \end{cases}$$

1) by Table

Example 8. $F(s) = \frac{2s+3}{s^2+9}$

$$\frac{2s}{s^2+9} + \frac{3}{s^2+9} = 2\,cos3t + sin3t$$

Example 9. $F(s) = \frac{s^2+2s}{s^4}$

$$\frac{s^2}{s^4} + \frac{2s}{s^4} = t + t^2$$

2) Laplace Transformation Theorem

Example 10. $F(s) = \frac{1}{s(s^2+1)} \rightarrow sinat \rightarrow sint$

$$= \int_0^t sint\ dt = [-cost]_0^t = -cost + \cos(0)$$

$$f(t) = 1 - cost$$

Example 11. $F(s)= \frac{1}{s^2+4s+5}$

$$F(s) = \frac{1}{s^2+4s+4+1} = \frac{1}{(s+2)^2+1}$$

$= \text{sint}.\,e^{\pm at} = sint.\,e^{-2t}$

Example 12. $F(s) = \frac{s}{s^2+4s+5}$

$$F(s) = \frac{s}{s^2 + 4s + 4 + 1} = \frac{s}{(s + 2)^2 + 1}$$

$$\frac{(s + 2)}{(s + 2)^2 + 1} - \frac{2}{(s + 2)^2 + 1} \rightarrow F(t) = cost\, e^{-2t} - 2sint. e^{-2t}$$

Application of Laplace transformation to solve Second Order Differential Equation

Four Steps to solve this case

1- Convert the equation to Laplace transform for example
$y" = s^2 x' - sx_o - x_1$, $y' = sx' - x_o$, $x = x'$
2- Insert the initial condition
3- Rearrange to obtain x'
4- Partial fraction

Example 13. Solve the equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t}$ $given\ that\ t = 0, x = 5, \frac{dx}{dt} = 7$

$$Sol:. (s^2 x' - sx_\circ - x_1) - 3(s x' - x_\circ) + 2 x' = \frac{2}{s-3}$$

$$s^2 x' - 5s - 7 - 3sx' + 15 + 2 x' = \frac{2}{s-3}$$

$$x'(s^2 - 3s + 2) - 5s + 8 = \frac{2}{s-3}$$

$$x' (s - 1)(s - 2) = \frac{2}{s-3} + 5s - 8$$

$$x' (s - 1)(s - 2) = \frac{2 + 5s (s - 3) - 8 (s - 3)}{s - 3}$$

$$x' = \frac{5 s^2 - 23s + 26}{(s - 3)(s - 1)(s - 2)}$$

$$\frac{5s^2 - 23s + 26}{(s - 1)(s - 2)(s - 3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$5 s^2 - 23s + 26 = A (s - 2)(s - 3) + B(s - 1)(s - 3) + c(s - 1)(s - 2)$
$5 s^2 - 23s + 26 = A(s^2 - 3s - 2s + 6) + B(s^2 - 3s - s + 3) + C(s^2 - 2s - s + 2)$
$5s^2 - 23s + 26 = A s^2 - 5As + 6A + Bs^2 - 4Bs + 3B + Cs^2 - 3Cs + 2C$
$5s^2 = A s^2 + B s^2 + C s^2 \rightarrow A + B + C = 5$ 1
$-23s = -5As - 4Bs - 3Cs$
$-23 = -5A - 4B - 3C$ 2
$26 = 6A + 3B + 2C$ 3
A=4, B= 0, C=1

$$x' = \frac{4}{s-1} + 0 + \frac{1}{s-3}$$

$$x = 4 e^t + e^{3t}$$