Biomedical Engineering
Engineering Analysis
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17/11/2021

Lecture (4)

Example. prove that Γ(n+1)= n Γ(n)

Sol.

$$\Gamma(n+1)=\int_0^\infty e^{-x}\,x^{n+1-1}.\,dx=\int_0^\infty e^{-x}\,x^{n}.\,dx$$

$$=\left[-x^n e^{-x}+n\int_0^\infty x^{n-1}.\,e^{-x}.\,dx\right]=[(-\infty e^{-\infty}-0)+n\Gamma(n)]=n\Gamma(n)$$

Multiplication of Two Gamma Functions

1) $\Gamma(\mathrm{n})\Gamma\left(\mathrm{n}+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma(2\mathrm{n})}{2^{2\mathrm{n}-1}}$
2) $\Gamma(\mathrm{p})\Gamma(1-\mathrm{p})=\frac{\pi}{\sin(\mathrm{p}\pi)}$

Example. Find the solve

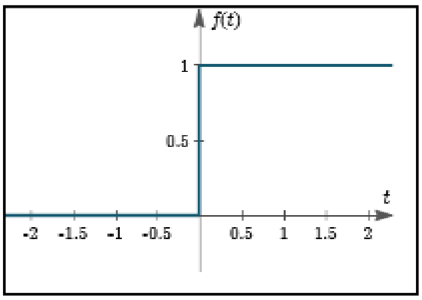
$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{\pi}{\sin\pi/3}=\frac{2\pi}{\sqrt{3}}$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\sqrt{\pi}\,\Gamma\frac{1}{2}}{1/\sqrt{2}}=\pi\sqrt{2}$$

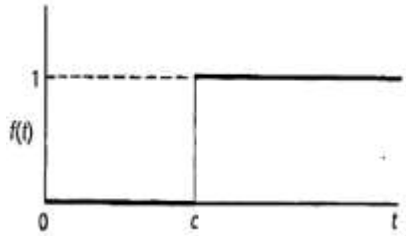
Unit Step Function (Heaviside Function)

The unit step function, u(t), is defined as That is, **u is a function of time t**, and u has value zero at all points to the left of the origin and is equal to one on the right of the origin. It is defined as:

$$\mathrm{U(t)}=\begin{cases}0, & \mathrm{t}<0\\ 1, & \mathrm{t}\geq 0\end{cases}$$



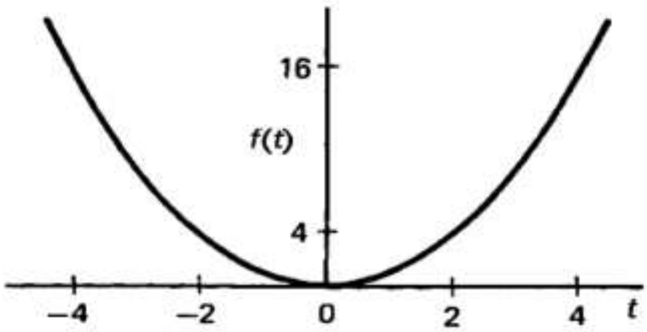
The displaced unit step function U(t-c) is defined as :



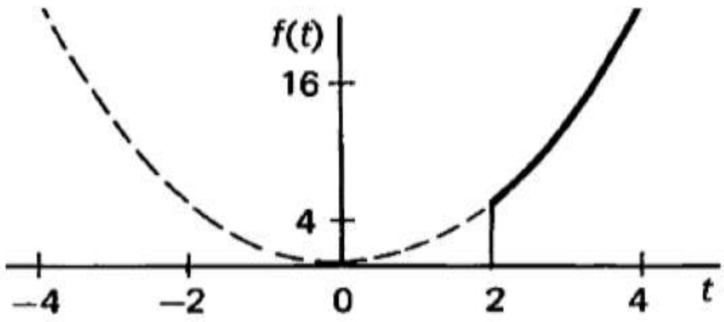
$$U(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

Effects of the unit step function

The graph of $f(t) = t^2$ as shown



The graph of $f(t) = u(t-2).t^2$



For $t < 2, u(t-2) = 0 \therefore u(t-2).t^2 = 0.t^2 = 0$

$t \geq 2, u(t-2) = 1 \therefore u(t-2).t^2 = 1.t^2 = t^2$

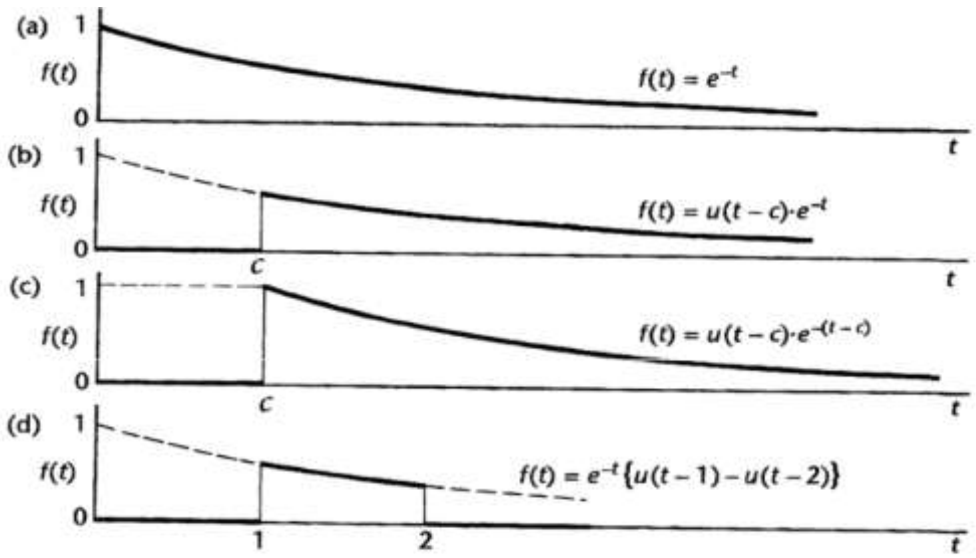
Example . Sketch the graph of the following functions for t> 0

a) $f(t) = e^{-t}$

b) $f(t) = u(t-c).e^{-t}$

c) $f(t) = u(t-c).e^{-(t-c)}$

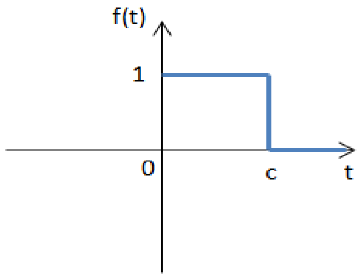
d) $f(t) = e^{-t}\{u(t-1) - u(t-2)\}$



Example. Sketch the graph of the function f(t)=1-U_c(t)

Sol.

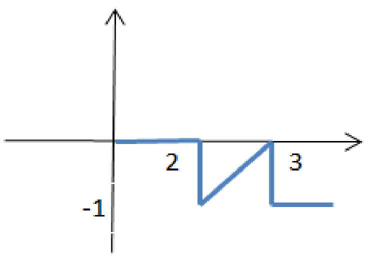
$$f(t) = 1 - \begin{cases} 0 & t < c \\ 1, & t \geq c \end{cases} = \begin{cases} 1 & t < c \\ 0 & t \geq c \end{cases} \qquad c \geq 0$$



Example. Sketch the graph of the function

y= (t-3) u_2(t) - (t-2) u_3(t) on the interval t ≥ 0

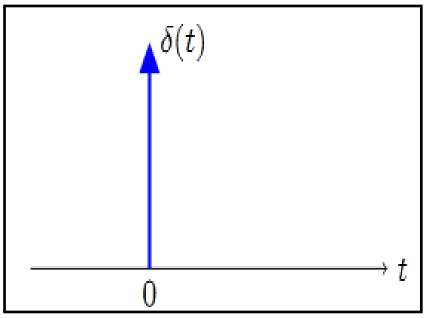
$$y = \begin{cases} (t-3)0 - (t-2)0, & t < 2 \\ (t-3)1 - (t-2)0, & 2 \leq t < 3 \\ (t-3)1 - (t-2)1, & t \geq 3 \end{cases} = \begin{cases} 0 & t < 2 \\ t-3 & 2 \leq t < 3 \\ -1 & t \geq 3 \end{cases}$$



Unit Impulse Function (Dirac delta function)

Dirac delta function (**δ function**), also known as the **unit impulse** symbol is a generalized function or distribution over the real numbers, whose value is zero everywhere except at zero, and whose integral over the entire real line is equal to one.

$$\delta(x) = \begin{cases} +\infty & , x = 0 \\ 0 & , x \neq 0 \end{cases}$$



Properties of the Dirac Delta Function

1. $\int_{-\infty}^{\infty} \delta(x)dx = 1$
2. $\delta(x) = \frac{d}{dx}u(x), where\ u(x) is\ the\ unit\ step\ function$
3. $\int_{-\infty}^{\infty} \delta(g(x))\, f(x)dx = \delta\big(g(x)\big) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$
4. $\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0)$
5. $\int_{-\infty}^{\infty} f(x)\delta(x - x_\circ)dx = f(x_\circ)$

Example . solve

$$\int_{-\infty}^{\infty} \delta\,(x^3 - x)e^{-x}dx$$

Sol.

$g(x) = x^3 - x\ ,\ g'(x) = 3x^2 - 1$

$g(x) = x(x^2 - 1) = x(x+1)(x-1) \quad ,x_1 = 0, x_2 = -1, x_3 = 1$

$$\delta(x^3 - x) = \frac{\delta(x)}{|-1|} + \frac{\delta(x+1)}{|2|} + \frac{\delta(x-1)}{|2|}$$

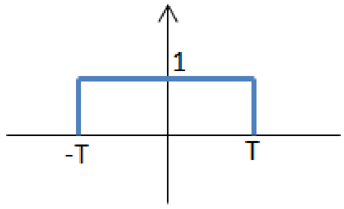
$$\int_{-\infty}^{\infty} \delta(x) + \frac{\delta(x+1)}{2} + \frac{\delta(x-1)}{2}\ e^{-x}dx = 1 + \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

Fourier Transform for Special Function

f(t)	F(ω)	f(t)	F(ω)
$e^{-at}, \quad t \geq 0$	$\frac{1}{i\omega + a}$	$1 - \frac{\lvert t\rvert}{T}\ , \lvert t\rvert < T$	$\frac{\sin^2(\frac{\omega T}{2})}{\omega^2 T/2}$
$t.e^{-at}$	$\frac{1}{(i\omega + a)^2}$		
1	$\delta(\omega)$	$e^{-t^2/2}$	$e^{-\omega^2/2}$
$\delta(t)$	1		
$\frac{a}{t^2+a^2}$	$e^{-a\lvert\omega\rvert}$	Sin (at) , $t > 0$	$\frac{a}{(i\omega)^2 + a^2}$
$e^{-a\lvert t\rvert}$	$\frac{a}{\omega^2 + a^2}$	Cos (at), $t > 0$	$\frac{i\omega}{(i\omega)^2 + a^2}$
U(t) , $\lvert t\rvert < T\ pulse$	$\frac{\sin(\omega T)}{\omega}$	Sin (at) $t \in (-\infty, \infty)$	$\frac{1}{2i}[\delta(\omega - a) - \delta(\omega + a)]$
		Cos(at) $t\in (-\infty, \infty)$	$\frac{1}{2}[\delta\,(\omega - a) + \delta(\omega + a)]$

Example . Find Fourier transform for f(t)=U(t) with |t|$< T$ where U(t) is the unit function

Sol.



t$< T, t > -T$

$$F(\omega) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} f(t).\cos(\omega t)\,.dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{T} \cos(\omega t)dt = [\frac{2}{\sqrt{2\pi}}\, sin\frac{\omega t}{\omega}]_0^T$$

$$= \frac{2}{\sqrt{2\pi}} \left[sin\frac{\omega T}{\omega}\right] - 0 = \frac{2}{\sqrt{2\pi}} \sin\frac{\omega T}{\omega}$$

Properties of Fourier Transform

1) Time Shift Law

f(t-a) u(t-a) $\longrightarrow (e^{-ia\omega}\, F(\omega))$

Example . Find Fourier transform for f(t) = $e^{-3t}.\mathrm{u}(\mathrm{t}-1)$

Sol. $e^{-3[(t-1)+1]}.u(t-1) = e^{-3-3(t-1)}.u(t-1)$

$\mathrm{e}^{-3}.e^{-3(t-1)}.u(t-1)$

$F(\omega) = e^{-3}\, e^{-3i\omega}.\, [\frac{1}{i\omega+3}]$

2) Frequency Shift (Modulation)

$$f(t).e^{\pm iat} \to F(\omega \pm a)$$

$$f(t).\cos(at) \to \frac{1}{2}\,[F(\omega - a) + F(\omega + a)]$$

$$f(t).\,\mathrm{Sin\ (at)} \to \frac{1}{2i}\,[F(\omega - a) - F(\omega + a)]$$

Example. Find F.T. for $f(t) = e^{-3|t|}.\cos(4t)$

Sol.

$$e^{-3|t|} \to \frac{3}{\omega^2 + 9}$$

$$F(\omega) = \frac{1}{2}\left[\frac{3}{(\omega - 4)^2 + 9} + \frac{3}{(\omega + 4)^2 + 9}\right]$$

Example . Find F.T. for $f(t) = U(t).\sin(5t)$

Sol.

$$U(t) \to \frac{\sin(\omega T)}{\omega}$$

$$U(t).\sin(5t) \to \frac{1}{2i}\left[\frac{\sin((\omega-5)T)}{(\omega-5)} - \frac{\sin((\omega+5)T)}{(\omega+5)}\right]$$

3) Derivative in time domain

$$\text{if } f(t) \to F(\omega)$$

$$\frac{df(t)}{dt} \to (i\omega)F(\omega)$$

$$\frac{d^2f(t)}{dt^2} \to (i\omega)^2\, F(\omega)$$

Example . Find F.T. for $f(t) = \frac{d^2}{dt^2}\,(t.e^{-4t})$

Sol.

$$t.e^{-4t} \to \frac{1}{(i\omega+4)^2} \quad , \qquad \frac{d^2}{dt^2}(t.e^{-4t}) \to (i\omega)^2.\left[\frac{1}{(i\omega+4)^2}\right]$$

4) Integration in time

$$\int_{-\infty}^{t} f(t)dt \to \left[\frac{F(\omega)}{(i\omega)}\right] + [\pi\, f(0)\delta(\omega)]$$

Example. Find F.T. for $f(t) = \int_{-\infty}^{t} e^{-4|t|}\,.dt$

Sol.

$$e^{-4|t|} \to \frac{4}{\omega^2+16}$$

$$\int_{-\infty}^{t} e^{-4|t|}\,dt \to \frac{1}{i\omega}\left[\frac{4}{\omega^2+16}\right]$$