Engineering Analysis

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Lecture (2)

Example1. Find the Fourier series of the function

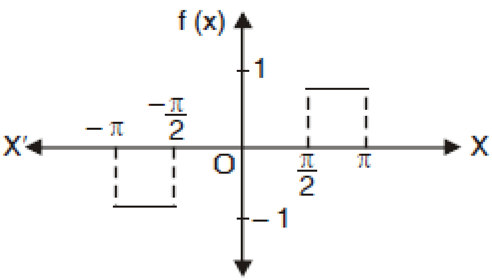
$$f(x)=\begin{cases}-1 & for\ -\pi < x < -\pi/2 \\ 0 & for\ -\frac{\pi}{2} < x < \pi/2 \\ 1 & for\ \ \pi/2 < x < \pi\end{cases}$$

Sol.

$f(x)= a_o+a_1 \cos x+a_2 \cos 2x+.......+a_n \cos nx+ b_1 \sin x+b_2 \sin 2x+ ....+ b_n \sin nx$

$a_o=1/2\pi\int_{-\pi}^{\pi} f(x)dx = 1/2\pi \int_{-\pi}^{-\pi/2}(-1)dx + 1/2\pi \int_{-\pi/2}^{\pi/2} 0\ dx + 1/2\pi \int_{\pi/2}^{\pi} 1\ dx$

$= 1/2\pi\ [\pi/2-\pi-\pi/2]=0$



$a_n= 1/\pi \int_{-\pi}^{\pi} f(x) cosnx\ dx$

$=1/\pi\int_{-\pi}^{-\pi/2}(-1) cosnx\ dx + 1/\pi \int_{-\pi/2}^{\pi/2} 0\ cosnx\ dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} 1\ cosnx\ dx$

$=-1/\pi \left[\frac{-\sin n\pi/2}{n} + \frac{\sin n\pi}{n}\right] + \frac{1}{\pi}\left[\frac{\sin n\pi}{n} - \frac{\sin n\pi/2}{n}\right] = 0$

$b_n= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\ dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1) \sin nx\ dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0) \sin nx\ dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1) \sin nx\ dx$

$= \pi \left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$

$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] - \frac{1}{n\,\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \frac{2}{n\,\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$

$b_1 = \frac{2}{\pi}$ $\qquad b_2 = -\frac{2}{\pi}$ $\qquad b_3 = \frac{2}{3\pi}$

$f(x)= \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \cdots\right]$

2.1. Ordinary Differential Equations

There are two types of second ordinary differential equations : homogenous and non-homogenous.

A second-order ODE can be written

$$y'' + 2ay' + by = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

The distinctive feature of this equation is that it is linear in y and its derivatives, whereas the function f on the right may be any given functions of x. If the equation begins with, f(x)y'' , then divide to have the **standard form** (1) with as the first term.

If f(x)=0 then ODE is called **homogeneous** such as:

$$y'' + 2ay' + by = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad 2$$

If $f(x) \neq 0$ then ODE is called **nonhomogeneous** such as Eq.1

2.1.1. Ordinary Differential Equation (homogeneous)

The nature of the solutions of ODE depends on the nature of the roots. There are three cases to consider:

1) Roots real and equals $r_1 = r_2$, $y = c_1 e^{rx} + c_2 x e^{rx}$
2) Roots real but not equal $r_1 \neq r_2$, $y = c_1 e^{r1x} + c_2 e^{r2x}$
3) Complex or imaginary $r = \alpha \pm i\beta$, $y = e^{\alpha x}[A \cos\beta x + B \sin\beta x]$

Example 2: Solve the ODE, $y'' + y' - 2y = 0$

Sol.

$$D^2 + D - 2 = 0$$

$$r^2 + r - 2 = 0$$

$$(r+2)(r-1) = 0$$

$$r = -2,\ r = 1 \longrightarrow y = c_1 e^{r1x} + c_2 e^{r2x} \longrightarrow y = c_1 e^{-2x} + c_2 e^{x}$$

Example 3: solve the ODE $y'' + 2y' + 2y = 0$

Sol.

$$D^2 + 2D + 2 = 0$$

$$r^2 + 2r + 2 = 0$$

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = -1 \pm i$$

$$y = e^{-x}\,[A \cos(x) + B \sin(x)]$$

2.1.2. Ordinary Differential Equation (nonhomogeneous)

The solution of nonhomogeneous ordinary differential equation (second order) is

$$y_{total}= y_c +y_p$$

to find y_c value (solution of the left side of equation)

to find y_p value (solution of the right side of equation)

Example 4. find the solution of ODE , y''-y'= 2 sinx

Sol.

$y_{total}=y_c+y_p$

$y_c \rightarrow D^2-D=0 \rightarrow r^2-r=0 \rightarrow r(r-1)=0 \rightarrow r=0, r=1$

$yc= c_1e^{r1x}+c_2e^{r2x} \rightarrow y_c= c_1e^0 + c_2e^x$

$y_p= A \sin x+ B \cos x \rightarrow y'_p= A\cos x- B \sin x \rightarrow y_p''= -A \sin x- B \cos x$

y''-y'=2 sinx

(-A sinx- B cosx)-(A cosx-B sinx)= 2sinx

-A sinx+Bsinx= 2 sinx $\rightarrow$ -A+B=2 …… 1

-B cox – A cosx=0 $\rightarrow$ A= -B sub in (1) $\rightarrow$ 2B=2

B=1, A= -1

$y_p= - \sin x +\cos x$

$y_{total=}$ $(c_1e^0+c_2e^x)+ (- \sin x+ \cos x)$

Example 5. Find the solution of ODE, 2y''-11y'+12y=3x-2

Sol. $y_{total} = y_c+ y_p$

$y_c \rightarrow 2D^2-11D+12=0$

$2r^2-11r+12=0$

$(2r-3)(r-4)= 0 , r_1=3/2, r_2=4$

$y_c= c_1e^{3/2x}+ c_2 e^{4x}$

$y_p= Ax+B \rightarrow y'_p= A \rightarrow y_p''= 0$

2(0)-11(A)+12 (Ax+B)= 3x-2

-11 A+12 Ax+12 B=3x-2

-11A+12B=-2 …………(1)

12Ax=3x $\rightarrow$ A= 3/12=1/4 sub in (1)

-11*1/4+12B= -2 $\rightarrow$ B= 1/16

$y_{p=}$ 1/4 x+1/16

$y_{total} = c_1 e^{3/2x}+c_2e^{4x}+ 1/4 x+1/16$

Example 6. Find the solution of ODE, $y''-6y'+9y= e^{3x}$

Sol.

$y_{total} = y_c + y_p$

$y_c \longrightarrow D^2-6D+9=0$

$r^2-6r+9= 0$

$(r-3)(r-3)=0$, $r_1=3$, $r_2=3$

$y_c= c_1e^{3x}+c_2xe^{3x}$

$y_p= Ax^2e^{3x}$, $y'_p= Ax^2\, 3e^{3x} + 2Axe^{3x}$, $y''_p= (9Ax^2e^{3x}+6Axe^{3x})+ (6Axe^{3x}+ 2Ae^{3x})$

$y_p''= (9Ax^2e^{3x}+ 12\, Axe^{3x}+ 2Ae^{3x})$

$(9Ax^2e^{3x}+ 12\, Axe^{3x}+ 2Ae^{3x}) – 6\, (Ax^2\, 3e^{3x} + 2Axe^{3x}) + 9\, (Ax^2e^{3x})= e^{3x}$

$2Ae^{3x} = e^{3x}$, $A = 1/2$

$y_p= 1/2\, x^2\, e^{3x}$

$y_t= c_1e^{3x}+c_2xe^{3x}+ 1/2\, x^2\, e^{3x}$

2.2. Application of Fourier Series to solve Ordinary Differential Equations (ODEs)

Example 7. Calculate the solution of equation

$y''+50y= f(x)$ where $f(x) = \left\{\begin{cases} 1 & for\ \ 0 < x < \pi \\ 0 & for\ \ x = 0,\ \pi\ to\ 2\pi \\ -1 & for\ \ \pi < x < 2\pi \end{cases}\right\}$ with $f(x)= f(x+2\pi)$ for all x

Sol.

$y''+ 50\, y =0$

$D^2+50=0$

$r^2+50=0$, $r^2= -50$, $r= \pm i\sqrt{50}$

$y_c= A \cos\sqrt{50}\, x+ B \sin \sqrt{50}x$

$y_p= f(x)$

$a_o= 1/2\pi \int_0^{\pi} 1\, dx + \int_{\pi}^{2\pi} -1\, dx$, $a_o= 0$

$a_n= 0$

$b_n= 1/\pi \int_0^{\pi} 1. \sin nx + \int_{\pi}^{2\pi} -1. \sin nx$

$b_n= 1/\pi\, [-1/n \cos nx]_0^{\pi}+ [+1/n \cos nx]_{\pi}^{2\pi}$

$b_n= 4/n\pi$

$f(x)= a_o+ \sum_1^{\infty} a_n\, cosnx + b_n\, sinnx$

$f(x)= \sum_1^{\infty} \frac{4}{n\pi} \sin nx$

$y_p= \sum_1^{\infty} 4/n\pi \sin nx \longrightarrow y_p= \sum_1^{\infty} C.sinnx$

$y'_p = \sum_1^\infty n.C.cosnx \longrightarrow y''_p = -\sum_1^\infty n^2.C.sinnx$

y''+50y = f(x)

$-\sum_1^\infty n^2.C.sinnx + 50\sum C.sinnx = \sum_1^\infty \frac{4}{n\pi} sinnx$

$C[-n^2 + 50] = 4/n\pi$

$C = 4/[-n^2+50].n\pi$

$y_p = \sum_1^\infty \frac{4}{[-n^2+50]n\pi}.sinnx$

$y_t = y_c + y_p$

$y_t = A\cos\sqrt{50} + B\sin\sqrt{50} + \sum_1^\infty \{4/[-n^2+50].n\pi\}.sinnx$

Homework

1) Solve the following Ordinary Differential Equations (ODEs)
   a) 3y''+4y-4=0
   b) 3y''-6y+3=0
   c) y''+3y+5=0
   d) y''+6y'+10y=0
   e) $y''+2k^2y'+k^4y=0$

2) Construct the Fourier series over the interval $-2 \le x \le 2$ for the function

$$f(x) = \begin{cases} 2 & -2 \le x \le 0 \\ x & 0 \le x \le 2 \end{cases}$$

3) Compute the Fourier series for $f(x) = x^2$ over the interval $-\pi \le x \le \pi$