Biomedical Engineering
Engineering Analysis
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Lecture (1)

Fourier Series

Fourier series: The Fourier series formula gives an expansion of a periodic function f(x) in terms of an infinite sum of sines and cosines . It is used to decompose any periodic function or periodic signal into the sum of a set of simple oscillating functions, namely sines and cosines.

- The two types of Fourier series formulas are Trigonometric series and exponential series formula.

Fourier series Formula:

$$y = f(x) = a_\circ + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$f(x) = a_\circ + a_1 \cos x + a_2 \cos 2x + \cdots \ldots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots \ldots . + b_n \sin nx$$

Fourier Series Coefficients:

$$a_\circ = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

To find Fourier Series Coefficients

$$y = f(x) = a_\circ + a_1 \cos x + a_2 \cos 2x + \cdots \ldots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x + \cdots \ldots + b_n \sin nx$$

$$f(x) = a_\circ + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$1)\int_0^{2\pi} f(x) = \int_0^{2\pi} a_\circ\, dx + \int_0^{2\pi} a_1\, \cos x\, dx + \cdots \ldots.. + \int_0^{2\pi} a_n\, \cos nx\, dx$$

$$+ \int_0^{2\pi} b_1 \sin x\, dx + \cdots \ldots. + \int_0^{2\pi} b_n \sin nx\, dx$$

$$\int_0^{2\pi} f(x)dx = a_\circ\,[2\pi] \rightarrow a_\circ = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$2)\int_0^{2\pi} f(x)\cos nx\, dx = \int_0^{2\pi} a_\circ \cos nx\, dx + a_1 \int_0^{2\pi} \cos x \cos nx\, dx + \int_0^{2\pi} a_2 \cos 2x \cos nx\, dx + \cdots \ldots +$
$\int_0^{2\pi} a_n \cos nx \cos nx\, dx + b_1 \int_0^{2\pi} \sin x \cos nx\, dx + b_2 \int_0^{2\pi} \sin 2x \cos nx\, dx +$
$\cdots \ldots.. + b_n \int_0^{2\pi} \sin nx \cos nx\, dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$\int_0^{2\pi} f(x) \sin nx\, dx = \int_0^{2\pi} a_\circ\, \sin nx\, dx + \int_0^{2\pi} a_1 \sin nx \cos nx\, dx + \int_0^{2\pi} a_2\, \cos 2x \sin nx\, dx + \cdots \ldots \ldots$$

$$+ a_n \int_0^{2\pi} \cos nx \sin nx\, dx + b_1 \int_0^{2\pi} \sin x \sin nx\, dx + b_2 \int_0^{2\pi} \sin 2x \sin nx\, dx + \cdots \ldots$$

$$+ b_n \int_0^{2\pi} \sin nx \sin nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx$$

Fourier series: Even & Odd Functions

- If f(x)= f(-x) for all x (Even Function)
- If – f(x)= f(-x) for all x (Odd Function)
- For all odd function

$$\int_{-L}^{L} f(x)dx = 0$$

- For all Even function

$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$$

Also: Odd function* Odd function = Even function

Even function* Even function = Even function

Odd function * Even function = Odd function

- If the function is an even then

$$a_\circ = \frac{1}{L}\int_{0}^{L} f(x)dx \qquad , a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad , \; b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\,, \;\; b_n = 0$$

$$f(x) = a_\circ + \sum_{n=1}^{\infty} a_n\, cos\frac{n\pi x}{L}$$

- If the function is an Odd function then

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \qquad , a_n = 0, \qquad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Complex Form of Fourier series

$$C_\circ = \frac{1}{2\pi}\int_{0}^{\pi} dx \qquad , \qquad C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \;\; , \quad f(x) = \sum_{n=-\infty}^{\infty} C_n\, e^{inx}$$

Half Range Fourier Series

Odd and Even Functions law use to solve the examples of Half range Fourier series

Example1: Find the Fourier series representing $f(x) = x \quad 0 < x < 2\pi$

Sol.:

$$f(x) = a_\circ + a_1 cosx + a_2\, cos2x + \cdots \ldots \ldots + b_1\, sinx + b_2\, sin2x$$

$$a_\circ = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x)cosnx\,dx = \frac{1}{\pi}\int_{0}^{2\pi} x\,cosnx\,dx = \frac{1}{\pi}\left[\left(x\,\frac{sinnx}{n} - 1\right) * \left(-\frac{cosnx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{cos2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) sinnx\, dx = \frac{1}{\pi}\int_0^{2\pi} x\, sinx\, dx = \frac{1}{\pi}\left[\left(x * -\frac{cosnx}{n} - 1\right) * \left(-\frac{sinnx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi\, cos2n\pi}{n}\right] = \frac{-2}{n}$$

$$x = \pi - 2\left[sinx + \frac{1}{2}\, sin2x + \frac{1}{3}\, sin3x + \cdots \ldots .\right]$$

$Example$ 2 : Find the Fourier series expansion of the periodic function of period -2π defined by

$$f(x) = \begin{Bmatrix} x & if\ \frac{-\pi}{2} < x < \frac{\pi}{2} \\ \pi - x & if\ \frac{\pi}{2} < x < \frac{3\pi}{2} \end{Bmatrix}$$

Sol.:

$$f(x) = \frac{a_\circ}{2} + a_1\, cosx + a_2\, cos2x + \cdots \ldots \ldots + b_1\, sinx + b_2\, sin2x$$

$$a_\circ = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\, dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x)dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{3\pi/2}$$

$$= \frac{1}{\pi}\left(\frac{\pi^2}{8} - \frac{\pi^2}{8}\right) + \frac{1}{\pi}\left(\frac{3\pi^2}{2} - \frac{9\pi^2}{8} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right) = \pi\left(\frac{3}{2} - \frac{9}{8} - \frac{1}{2} + \frac{1}{8}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\, cosnx\, dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x) cosnx\, dx$$

$$\frac{1}{\pi}\left[x\frac{sinnx}{n} - 1 * \frac{-cosnx}{n^2}\right]_{-\pi/2}^{\pi/2} + \frac{1}{\pi}\left[(\pi - x)\frac{sinnx}{n} - (-1)\left(\frac{-cosnx}{n^2}\right)\right]_{\pi/2}^{3\pi/2}$$

$$\frac{1}{\pi}\left[\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} - \frac{\cos\frac{n\pi}{2}}{n^2}\right] + \frac{1}{\pi}\left[\frac{-\pi}{2}\frac{\sin\frac{3n\pi}{2}}{n} + \frac{\cos\frac{3n\pi}{2}}{n^2} - \frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n} + \frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$\frac{1}{\pi}\left[\frac{-\pi}{2n}\left(\sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos\frac{3n\pi}{2} - \cos\frac{n\pi}{2}\right)\right] = \frac{1}{\pi}\left[\frac{-\pi}{n}\, sin\, n\pi \cos\frac{n\pi}{2} + \frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right]$$

$$= 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} x\, sinnx\, dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x) sinnx\, dx = \frac{2}{\pi}\int_0^{\pi/2} x \sin nx\, dx + \frac{1}{\pi}\int_{\pi/2}^{3\pi/2} (\pi - x) sinnx\, dx$$

$$=\frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)-1*\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2}+\frac{1}{\pi}\left[(\pi-x)\left(\frac{-\cos nx}{n}\right)-(-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{3\pi/2}$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]+\frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+3\frac{\sin\frac{n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$\frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\ \sin nx+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$=\frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2}-\sin\frac{3n\pi}{2}\right]$$

$$f(x)=\frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}\dots\dots\dots\right]$$

Example 3: Obtain the complex form of the Fourier series of the function

$$f(x)=\begin{cases}0 & -\pi\le x\le 0\\ 1 & 0\le x\le\pi\end{cases}$$

Sol.:

$$C_\circ=\frac{1}{2\pi}\int_0^{\pi}dx=\frac{1}{2}$$

$$C_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}\,dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0\,e^{-inx}\,dx+\int_0^{\pi}1\,e^{-inx}\,dx\right]=\frac{1}{2\pi}\int_0^{\pi}e^{-inx}\,dx=\frac{1}{2\pi}\left[\frac{e^{-inx}}{-in}\right]_0^{\pi}$$

$$-\frac{1}{2n\pi i}\left[e^{-in\pi}-1\right]=\frac{1}{2n\pi i}\left[\cos n\pi-i\sin n\pi-1\right]=\frac{-1}{2ni\pi}\left[(-1)^n-1\right]$$

$$=\begin{cases}\frac{1}{in\pi}, & n\text{ is odd}\\ 0, & n\text{ is even}\end{cases}$$

$$f(x)=\frac{1}{2}+\frac{1}{i\pi}\left[\frac{e^{ix}}{1}+\frac{e^{3ix}}{3}+\frac{e^{5ix}}{5}+\cdots\dots\right]+\frac{1}{i\pi}\left[\frac{e^{-ix}}{-1}+\frac{e^{-3ix}}{-3}+\frac{e^{-5ix}}{5}+\cdots\dots\right]$$

$$=\frac{1}{2}-\frac{1}{i\pi}\left[\left(e^{ix}-e^{-ix}\right)+\frac{1}{3}\left(e^{3ix}-e^{-3ix}\right)+\frac{1}{5}\left(e^{5ix}-e^{-5ix}\right)+\cdots\dots\right]$$

$$\frac{1}{2} + i\frac{1}{\pi}\left[\left[e^{ix} - e^{-ix}\right] + \frac{1}{3}\left(e^{3ix} - e^{3ix}\right) + \frac{1}{5}\left(e^{5ix} - e^{-5ix}\right)\right]$$