2.3 The Conduction Equation of Spherical Coordinates:

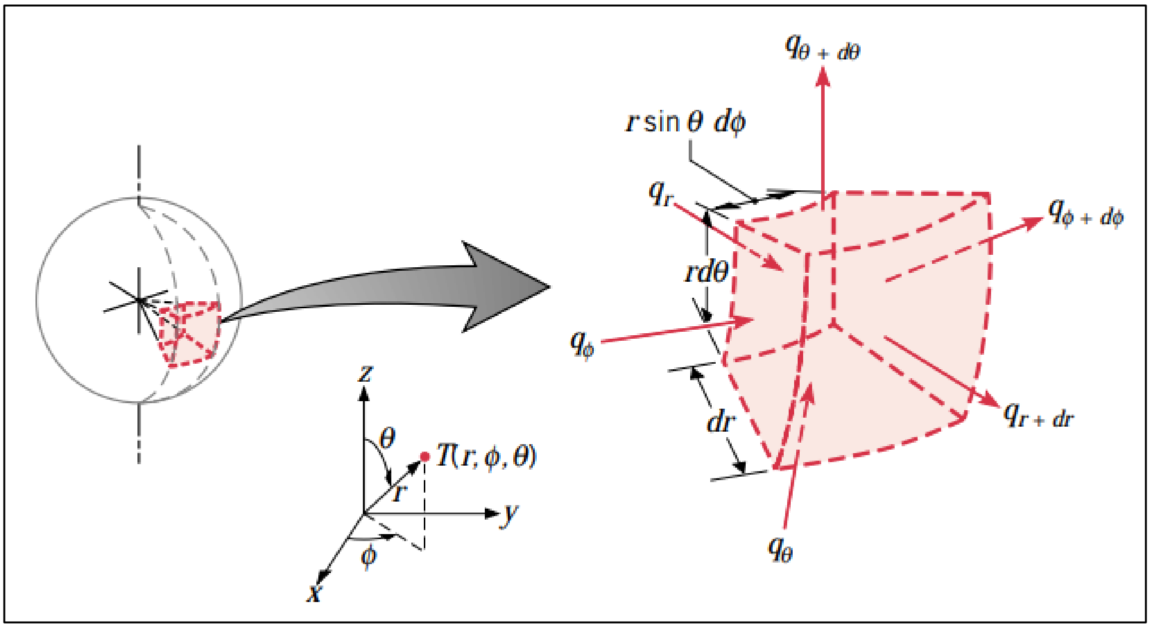


Figure (2.3) Differential Control Volume ($dr.rsin\theta d\emptyset.rd\theta$) for Conduction Analysis in Spherical Coordinates ($r,\theta,\emptyset$)

In spherical coordinates the general form of the heat flux vector and Fourier's law is

$$q_r'' = -k\frac{\partial T}{\partial r} \qquad (2.13a)$$

$$q_\theta'' = -\frac{k}{r}\frac{\partial T}{\partial \theta} \qquad (2.13b)$$

$$q_z'' = -\frac{1}{rsin\theta}\frac{\partial T}{\partial \emptyset} \qquad (2.13c)$$

Applying an energy balance to the differential control volume of Figure 2.3 the following general form of the heat equation is obtained:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \emptyset}\left(k\frac{\partial T}{\partial \emptyset}\right)+\frac{1}{r^2sin\theta}\frac{\partial}{\partial \theta}\left(ksin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho C_p\frac{\partial T}{\partial t} \qquad (2.14)$$

The following forms under the specific condition:

CASE (1): One dimension, unsteady-state and homogenous material (isotropic material).

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

CASE (2): one dimension, steady-state ($\partial/\partial t = 0$), homogenous material (isotropic material) and with heat generation.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{\dot{q}}{k}=0$$

CASE (3): one dimension, unsteady state, homogenous material (isotropic material) and without heat generation.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=\frac{1}{\alpha}\frac{\partial T}{\partial t}$$

CASE (4): one dimension, steady state ($\partial/\partial t = 0$), homogenous material (isotropic material) and without heat generation.

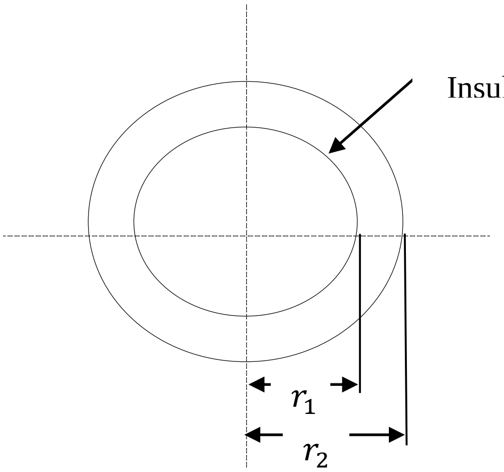
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)=0$$

The solution of the heat equation is:

$$q_r=\frac{4\pi k(T_1-T_2)}{\left(\frac{1}{r_1}-\frac{1}{r_2}\right)} \tag{2.15}$$

Example (2.5): A spherical container having outer diameter (500 mm) is insulated by (100 mm) thick layer of material with thermal conductivity ($k = 0.3(1 + 0.006T)$) W/m. °C, where T in °C. If the surface temperature of sphere is (-200 °C) and temperature of outer surface is (30 °C) determine the heat flow.

Solution:



$$q = \frac{4\pi k(T_1 - T_2)}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)}$$

$$r_1 = \frac{D}{2} = \frac{500}{2} = 250\ mm$$

$$r_2 = r_1 + 100 = 350\ mm$$

$$k = 0.3(1 + 0.006T) = 0.3\left(1 + 0.006\left(\frac{-200 + 30}{2}\right)\right)$$

$$k = 0.147\ W/m.°C$$

$$q = \frac{4\pi * 0.147(-200 - 30)}{\left(\frac{1}{0.25} - \frac{1}{0.35}\right)} = -371.53\ W$$

Example (2.6): A hollow sphere the inner and outer shell radius are (r_i) and (r_0) with uniform temperature at the inner and outer surface (T_i) and (T_0). Find the temperature distribution equation as a function of radius r without heat generation in the steady state.

Solution:

Assumption:

1- Steady state ($\partial/\partial t = 0$).
2- Homogenous material (isotropic material).
3- Without heat generation.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0 \qquad multiply\ by\ r^2$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = 0 \qquad integrate$$

$$r^2\frac{\partial T}{\partial r} = C_1$$

$$\frac{\partial T}{\partial r} = \frac{C_1}{r^2} \qquad (1) \quad integrate\ again$$

$$T = -\frac{C_1}{r} + C_2 \qquad (2)$$

B.C1: $at\ r = r_i \qquad T = T_i \qquad sub.in\ Eq.(2)$

$$T_i = -\frac{C_1}{r_i} + C_2 \qquad (3)$$

B.C2: $at\ r = r_0 \qquad T = T_0 \qquad sub.in\ Eq.(2)$

$$T_0 = -\frac{C_1}{r_0} + C_2 \qquad (4)$$

Subtract Eq. (3) and Eq. (4)

$$T_i - T_0 = C_1\left(\frac{1}{r_0} - \frac{1}{r_i}\right)$$

$$C_1 = \frac{T_i - T_0}{\frac{1}{r_0} - \frac{1}{r_i}} \qquad sub.in\ Eq.(3)$$

$$T_i = \frac{T_i - T_0}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)r_i} + C_2$$

$$C_2 = T_i - \frac{T_i - T_0}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)r_i}$$

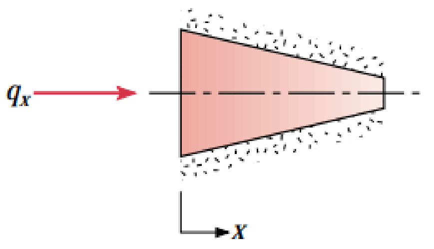
Sub. $C_1\ and\ C_2$ in Eq. (2)

$$T = -\frac{T_i - T_0}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)r} + T_i - \frac{T_i - T_0}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)r_i}$$

$$T = \frac{T_0 - T_i}{\left(\frac{1}{r_0} - \frac{1}{r_i}\right)}\left(\frac{1}{r} + \frac{1}{r_i}\right) + T_i$$

Home Work (2):

1- Assume steady-state, one-dimensional heat conduction through the symmetric shape shown. Assuming that there is no internal heat generation, derive an expression for the thermal conductivity $k(x)$ for these conditions: $A(x) = (1 - x)$, $T(x) = 300(1 - 2x - x^3)$, and $q = 6000\ W$, where A is in square meters, T in kelvins, and x in meters.

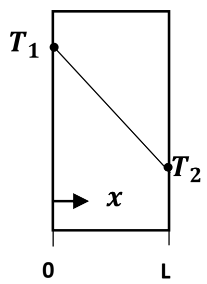


2- A plane wall is constructed of a material having a thermal conductivity that varies as the square of the temperature according to the relation $k = k_0(1 + \beta T^2)$. Derive an expression for the heat transfer in such a wall.

3- The temperature distribution across a wall 0.3 m thick at a certain instant of time is as $T(x) = a + bx + cx^2$ where T is in degrees Celsius and x is in meters, while $= 200\ °C$, $b = -200\ °C/m\ and\ c = 30\ °C/m^2$. The wall has a thermal conductivity of $k = 1\ W/m.K$. On a unit surface area basis, determine the rate of heat transfer into and out of the wall and the rate of change of energy stored by the wall.

4- A large thin concrete slab of thickness L. selling is an exothermic process that releases $\dot{q}(\frac{W}{m^3})$. The outside surfaces are kept at the ambient temperature so $T_\infty = T_w$. What is the maximum internal temperature?

5- A slab shown in figure is at steady state with dissimilar temperature on either side or no internal heat generation. Derive an expression for the temperature distribution and heat flux through it.



6- Consider a long solid tube, insulated at the outer radius (r_2) and cooled at the inner radius (r_1), with uniform heat generation ($\dot{q}(W/m^3)$) within the solid.

a- Obtain the general solution for the temperature distribution in the tube.
b- Determine the heat removal rate per unit length of tube.

7- Derive an expression for the temperature distribution in a sphere of radius (r_0) with uniform heat generation ($\dot{q}(W/m^3)$) and constant surface temperature (T_0).

8- A hollow sphere is constructed of aluminum (k=204 W/m. °C) with an inner diameter of (4 cm) and an outer diameter of 8 cm. The inside temperature is (100 °C) and the outer temperature is (50 °C). Calculate the heat transfer.

9- Derive an expression for the temperature distribution in a hollow cylinder with heat sources that vary according to the linear relation $\dot{q} = a + br$ with $\dot{q}_i$ the generation rate per unit volume at $r = r_i$. The inside and outside temperatures are $T = T_i$ at $r = r_i$ and $T = T_0$ at $r = r_0$.