2.2 The Conduction Equation of Cylindrical Coordinates:

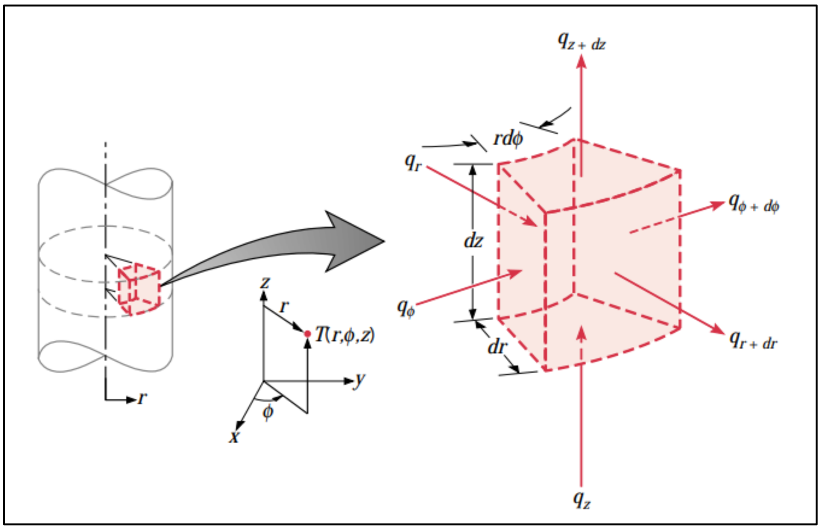


Figure (2.2) Differential Control Volume ($dr.r\ d\emptyset.dz$) for Conduction Analysis in Cylindrical Coordinates ($r,\emptyset,z$).

The general form of the heat flux vector, and hence of Fourier's law, is

$$q''_r = -k\frac{\partial T}{\partial r} \tag{2.10a}$$

$$q''_\emptyset = -\frac{k}{r}\frac{\partial T}{\partial \emptyset} \tag{2.10b}$$

$$q''_z = -k\frac{\partial T}{\partial z} \tag{2.10c}$$

Applying an energy balance to the differential control volume of Figure (2.2) the following general form of the heat equation is obtained:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \emptyset}\left(k\frac{\partial T}{\partial \emptyset}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho C_p\frac{\partial T}{\partial t} \tag{2.11}$$

The following forms under the specific condition:

CASE (1): One dimension and homogenous material (isotropic material).

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

CASE (2): one dimension, steady state ($\partial/\partial t = 0$), homogenous material (isotropic material) and with heat generation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\dot{q}}{k} = 0$$

CASE (3): one dimension, unsteady state, homogenous material (isotropic material) and without heat generation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

CASE (4): one dimension, steady-state ($\partial/\partial t = 0$), homogenous material (isotropic material) and without heat generation.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0$$

The solution of the heat equation is:

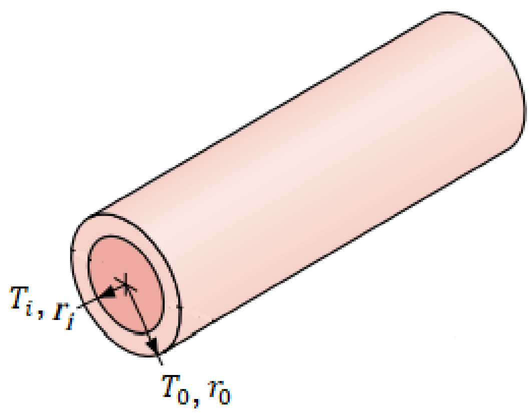
$$q = -kA\frac{\partial T}{\partial r} = -2\pi rLk\frac{(T_i - T_0)}{\ln\frac{r_i}{r_0}} = 2\pi rLk\frac{(T_i - T_0)}{\ln\frac{r_0}{r_i}} \qquad (2.12)$$

Example (2.3): consider a steam pipe of length (L), inner radius (r_i), outer radius (r_0) and thermal conductivity (k). The inner and outer surface of pipe are maintained at average temperature of (T_i) and (T_0) respectively. Obtain a general relation for the temperature distribution inside the pipe under steady conditions and determine the rate of heat loss from the steam through the pipe.

Solution:

Assumption:

1- Steady-state ($\partial/\partial t = 0$).
2- Homogenous material (isotropic material).
3- With heat generation.



$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \qquad \textit{multiply by } r$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \qquad \textit{integrate}$$

$$r\frac{\partial T}{\partial r}=C_1 \quad \rightarrow \quad \frac{\partial T}{\partial r}=\frac{C_1}{r} \qquad (1) \qquad integrate\ again$$

$$T=C_1\ln r+C_2 \qquad (2)$$

B.C1: $at\ r=r_i \qquad T=T_i \qquad sub.in\ Eq.(2)$

$$T_i=C_1\ln r_i+C_2 \qquad (3)$$

B.C2: $at\ r=r_0 \qquad T=T_0 \qquad sub.in\ Eq.(2)$

$$T_0=C_1\ln r_0+C_2 \qquad (4)$$

Subtract Eq. (3) and Eq. (4)

$$T_i-T_0=C_1\ln\frac{r_i}{r_0}$$

$$C_1=\frac{T_i-T_0}{\ln\frac{r_i}{r_0}} \qquad sub.in\ Eq.(3)$$

$$T_i=\frac{T_i-T_0}{\ln\frac{r_i}{r_0}}\ln r_i+C_2$$

$$C_2=T_i-\frac{T_i-T_0}{\ln\frac{r_i}{r_0}}\ln r_i$$

Sub. $C_1\ and\ C_2$ in Eq. (2)

$$T=\frac{T_i-T_0}{\ln\frac{r_i}{r_0}}\ln r+T_i-\frac{T_i-T_0}{\ln\frac{r_i}{r_0}}\ln r_i$$

$$T=\frac{T_i-T_0}{\ln\frac{r_i}{r_0}}\ln\frac{r}{r_i}+T_i$$

$$q=-kA\frac{\partial T}{\partial r}=-k(2\pi rL)\frac{C_1}{r}=-\frac{(2\pi rLk)}{r}\frac{T_i-T_0}{\ln\frac{r_i}{r_0}}$$

$$q=-2\pi Lk\frac{(T_i-T_0)}{\ln\frac{r_i}{r_0}}$$

Example(2.4): Uniform internal heat generation $\dot{q} = 5 \times 10^7\ W/m^3$ is occurring in a cylindrical nuclear reactor fuel rod of (50 mm) diameter, and under steady-state conditions, the temperature distribution is of the form $T(r) = a + br^2$, where T is in degrees Celsius and r is in meters, while $a = 800\ °C$ and $b = -4.167 \times 10^5\ °C/m^2$. The fuel rod properties are $k = 30\ W/m.K,\ \rho = 1100\ kg/m^3\ and\ C_p = 800\ J/kg\,.K.$

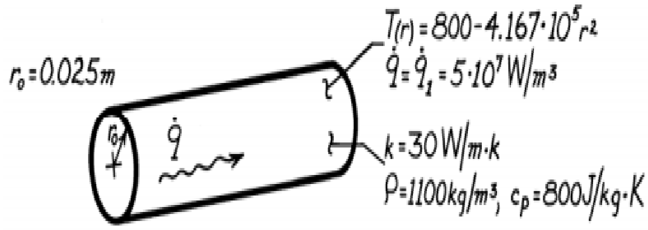
(a) What is the rate of heat transfer per unit length of the rod at $r = 0$ (the centerline) and at $r = 25\ mm$ (the surface)?

(b) If the reactor power level is suddenly increased to $\dot{q}_2 = 10^8\ W/m^3$, what is the initial time rate of temperature change at $r = 0$ and $r = 25\ mm$?

Solution:

Assumptions:

1- One-dimensional conduction in the r direction.
2- Uniform generation.
3- Steady-state for $\dot{q}_1 = 5 \times 10^7\ W/m^3$



(a) Steady-state centerline and surface heat transfer rates per unit length, (q'_r).

$$q''_r = -k\frac{\partial T}{\partial r} \qquad\qquad q_r = -kA_r\frac{\partial T}{\partial r}$$

$$q_r = -k(2\pi rL)\frac{\partial T}{\partial r}$$

$$q'_r = -k(2\pi r)\frac{\partial T}{\partial r}$$

at $r = 0$

$$\left.\frac{\partial T}{\partial r}\right]_{r=0} = 2br = o$$

$$\therefore q'_r = 0$$

at $r = r_0$

$$\left.\frac{\partial T}{\partial r}\right]_{r=r_0} = -2 \times 4.167 \times 10^5 r = -2 \times 4.167 \times 10^5 r_0$$

$$\left.\frac{\partial T}{\partial r}\right]_{r=r_0} = -2 \times 4.167 \times 10^5 \times 0.025$$

$$\left.\frac{\partial T}{\partial r}\right]_{r=r_0} = -0.208 \times 10^5 \; K/m$$

$$q'_r = -k(2\pi r_0)\frac{\partial T}{\partial r} = -30 * 2\pi * 0.025 * -0.208 \times 10^5 = 0.98 \times 10^5 W/m$$

(b) The initial time rate of change of the centerline and surface temperatures in response to a change in the generation rate from $\dot{q}_1 to\ \dot{q}_2 = 10^8 \frac{W}{m^3}$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \dot{q}_2 = \rho C_p \frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial t} = \frac{1}{\rho C_p}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \dot{q}_2\right] \qquad (1)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = \frac{k}{r}\frac{\partial}{\partial r}[r(-8.334 \times 10^5 r)] = \frac{k}{r}\frac{\partial}{\partial r}[-8.334 \times 10^5 r^2]$$

$$= \frac{k}{r}[-8.334 \times 10^5 \times 2r] = -16.668 \times 10^5 k = -16.668 \times 10^5 \times 30$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) = -5 \times 10^7 \frac{W}{m^3} \qquad sub.\ in\ Eq.(1)$$

$$\frac{\partial T}{\partial t} = \frac{1}{1100 \times 800}[-5 \times 10^7 + 10^8]$$

$$\frac{\partial T}{\partial t} = 56.82 \; K/s$$