CHAPTER TWO
Introduction to Conduction

Conduction refers to the transport of energy in a medium due to a temperature gradient, and the physical mechanism is one of random atomic or molecular activity.

2.1 The Conduction Equation of Rectangular Coordinate:

Consider an element of small control volume $dV = dx\,dy\,dz$ as shown in Figure (2.1) and the temperature distribution $T(x, y, z, t)$ is expressed in Cartesian coordinates where (t) is the time.

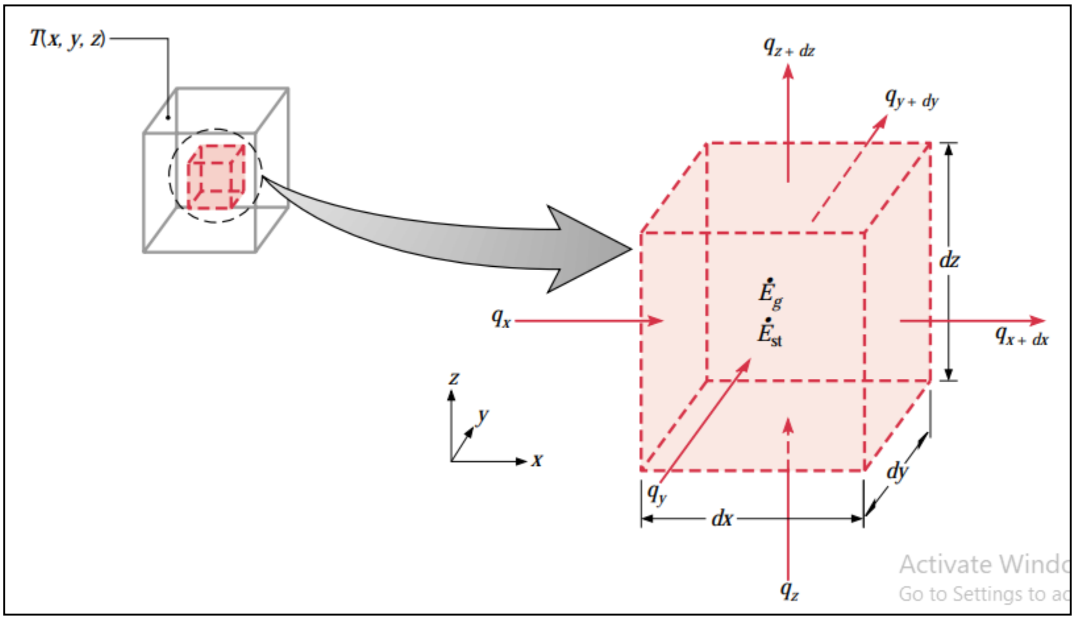


Figure (2.1) Differential Control Volume ($dx\,dy\,dz$) for Conduction Analysis in Cartesian Coordinates.

The conduction heat rates perpendicular to each of the control surfaces at the *x*, *y*, and *z* coordinate locations are indicated by the terms q_x, q_y, and q_z, respectively.

$$q_x = -kA\frac{\partial T}{\partial x} = -kdydz\frac{\partial T}{\partial x} \qquad (2.1a)$$

$$q_y = -kA\frac{\partial T}{\partial y} = -kdxdz\frac{\partial T}{\partial y} \qquad (2.1b)$$

$$q_z = -kA\frac{\partial T}{\partial z} = -kdxdy\frac{\partial T}{\partial z} \qquad (2.1c)$$

The conduction heat rates at the opposite surfaces can then be expressed as a Taylor series expansion where, neglecting higher order terms,

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x} dx \tag{2.2a}$$

$$q_{y+dy} = q_y + \frac{\partial q_y}{\partial y} dy \tag{2.2b}$$

$$q_{z+dz} = q_z + \frac{\partial q_z}{\partial z} dz \tag{2.2c}$$

Within the medium there may also be an energy source term associated with the rate of thermal energy generation. This term is represented as

$$\dot{E}_g = \dot{q}V = \dot{q}\, dx\, dy\, dz \tag{2.3}$$

where $\dot{q}$ is the rate at which energy is generated per unit volume of the medium (W/m^3).

In addition, there may occur changes in the amount of the internal thermal energy stored by the material in the control volume. If the material is not experiencing a change in phase, latent energy effects are not pertinent, and the energy storage term may be expressed as

$$\dot{E}_{st} = \dot{m}C_p \frac{\partial T}{\partial t} = \rho C_p V \frac{\partial T}{\partial t} = \rho C_p\, dx\, dy\, dz \frac{\partial T}{\partial t} \tag{2.4}$$

Where

$\rho C_p \frac{\partial T}{\partial t}$ is the time rate of change of the sensible (thermal) energy of the medium per unit volume.

C_p is specific heat capacity (J/kg. °C).

$\frac{\partial T}{\partial t}$ is the temperature change with time (K/s)

V is the volume (m^3).

ρ is the density (kg/m^3).

The general form of the conservation of energy requirement is

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st} \tag{2.5}$$

Substitute Eq. (2.1) to (2.4) in Eq. (2.5) get

$$\left(q_x + q_y + q_z\right) + \dot{q}\, dx\, dy\, dz - \left(q_{x+dx} + q_{y+dy} + q_{z+dz}\right) = C_p\, dx\, dy\, dz \frac{\partial T}{\partial t} \quad (2.6)$$

$$\dot{q}\, dx\, dy\, dz - \frac{\partial q_x}{\partial x} dx - \frac{\partial q_y}{\partial y} dy - \frac{\partial q_z}{\partial z} dz \partial t = \rho C_p\, dx\, dy\, dz \frac{\partial T}{\partial t} \quad (2.7)$$

$$\dot{q}\, dx\, dy\, dz - \frac{\partial}{\partial x}\left(-k dydz \frac{\partial T}{\partial x}\right) dx - \frac{\partial}{\partial y}\left(-k dxdz \frac{\partial T}{\partial y}\right) dy - \frac{\partial}{\partial z}\left(-k dxdy \frac{\partial T}{\partial z}\right) dz \partial t = \rho C_p\, dx\, dy\, dz \frac{\partial T}{\partial t} \quad (2.8)$$

Divided Eq. (2.8) by $(dx\, dy\, dz)$ get the general equation for conduction in a rectangular coordinate system

$$\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{q} = \rho C_p \frac{\partial T}{\partial t} \quad (2.9)$$

The following forms under the specific condition:

CASE (1): For homogenous material (isotropic material)

$k = constant$

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right) + \frac{\dot{q}}{k} = \frac{\rho C_p}{k} \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

Where $\alpha = \frac{k}{\rho C_p} = Thermal\ diffusivity\ (m^2/s)$

CASE (2): For steady-state $(\partial/\partial t = 0)$, homogenous material $(k = constant)$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = 0$$

CASE (3): For steady-state, without heat generation and homogenous material

$$\frac{\partial}{\partial t} = 0 \qquad k = constant \qquad \dot{q} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$

CASE (4): 2-Dimension, steady-state, homogenous material and without heat generation

$$\frac{\partial}{\partial t} = 0 \qquad \dot{q} = 0 \qquad k = constant \qquad \frac{\partial}{\partial z} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

CASE (5): One-Dimension, steady-state, homogenous material and without heat generation

$$\frac{\partial}{\partial t} = 0, \qquad \dot{q} = 0, \qquad \frac{\partial}{\partial y} = 0 \qquad \frac{\partial}{\partial z} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} = 0$$

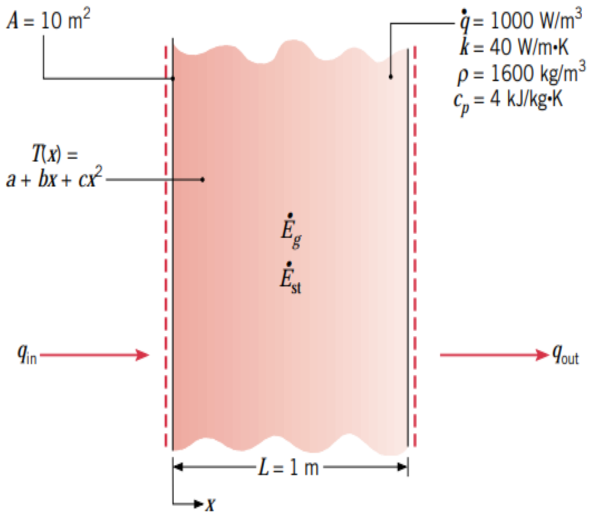
We may use Fourier's law to determine the conduction heat transfer rate. That is

$$q_x = -kA\frac{\partial T}{\partial x} = \frac{kA}{L}(T_1 - T_2)$$

Example (2.1): The temperature distribution across a wall (1m) thick at a certain instant of time is given as $T(x) = a + bx + cx^2$ where T is in degrees Celsius and x is in meters, while $= 900\ °C, b = -300\ °C/m\ and\ c = -50\ °C/m^2$. A uniform heat generation $\dot{q} = 1000\ W/m^3$ is present in the wall of area 10 m^2 having the properties $\rho = 1600\ kg/m^3$, $k = 40\ W/m.K$ and $C_p = 4\ kJ/kg.K$.

1. Determine the rate of heat transfer entering the wall ($x = 0$) and leaving the wall ($x = 1\ m$).
2. Determine the rate of change of energy storage in the wall.
3. Determine the time rate of temperature change at $x = 0, 0.25, and\ 0.5\ m$.

Solution:



Assumptions:

1- One-dimensional conduction in the x-direction.
2- Isotropic medium with constant properties.
3- Uniform internal heat generation, ($\dot{q} = 1000\ W/m^3$).

1- Heat rates entering q_{in} ($x = 0$) and leaving q_{out} ($x = 1\ m$) the wall.

$$q_{in} = -kA\frac{dT}{dx}\bigg|_{x=0} = -kA(b + 2cx)_{x=0}$$

$$q_{in} = -kAb$$

$$q_{in} = -40 * 10 * -300 = 120000\ W = 120KW$$

$$q_{out} = -kA\frac{dT}{dx}\bigg|_{x=L} = -kA(b + 2cx)_{x=L}$$

$$q_{out} = -kA(b + 2cL)$$

$$q_{out} = 40 * 10 * (-300 + 2 * -50 * 1)$$

$$q_{out} = 160000\ W = 160\ KW$$

2- Rate of change of energy storage in the wall $\dot{E}_{st}$.

$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$

Where $\dot{E}_g = \dot{q}V = \dot{E}_g = \dot{q}AL$

$\dot{E}_{st} = \dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{in} + \dot{q}AL - \dot{E}_{out}$

$\dot{E}_{st} = 120000 + 1000 * 10 * 1 - 160000$

$\dot{E}_{st} = -30000\, W = -30\, KW$

3- Time rate of temperature change at $x = 0, 0.25, and\ 0.5\ m$.

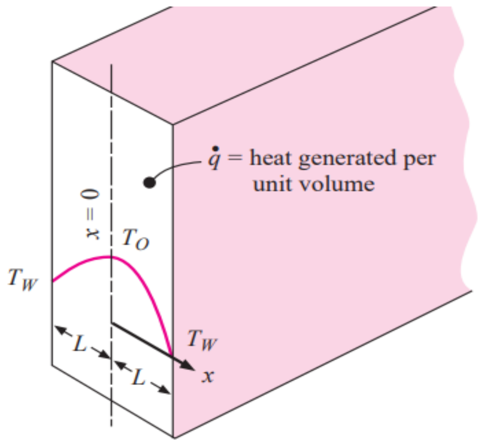
$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho C_p}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}(b + 2cx) = 2c = 2 * -50 = -100\ °C/m^2$$

$$\frac{\partial T}{\partial t} = \frac{40}{1600 * 4} * (-100) + \frac{1000}{1600 * 4}$$

$$\frac{\partial T}{\partial t} = -0.625 + 0.156 = -0.468\ °C/s \qquad for\ x = 0, 0.25\ and\ 0.5$$

Example (2.2): Consider the plane wall with uniformly distributed heat sources shown in Figure. The thickness of the wall in the x direction is 2L, and it is assumed that the dimensions in the other directions are sufficiently large that the heat flow may be considered as one dimensional. The heat generated per unit volume is $\dot{q}$, and assume that the thermal conductivity does not vary with temperature. Derive an expression of the temperature distribution.



Solution:

Assumption:

1- One-Dimension ($\frac{\partial}{\partial y} = 0, \frac{\partial}{\partial z} = 0$).

2- Steady state ($\frac{\partial}{\partial t} = 0$)

3- Uniform heat generation ($\dot{q}$).

4- Homogeneous ($k = constant$).

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = 0 \qquad integrate$$

$$\frac{\partial T}{\partial x} + \frac{\dot{q}}{k} x = C_1 \qquad (1) \quad integrate\ again$$

$$T + \frac{\dot{q}}{k} x^2 = C_1 x + C_2$$

$$T = -\frac{\dot{q}}{2k} x^2 + C_1 x + C_2 \qquad (2)$$

B.C1: $at\ x = 0$ $\qquad T = T_0$ Sub. in Eq. (2)

$$T_0 = -\frac{\dot{q}}{2k}(0)^2 + C_1 * 0 + C_2$$

$C_2 = T_0$ $\qquad$ Sub. in Eq. (2)

B.C2: $at\ x = \pm L$ $\qquad T = T_w$ Sub. in Eq. (2)

$$T_w = -\frac{\dot{q}}{2k} L^2 + C_1 L + T_0 \qquad (3)$$

$$T_w = -\frac{\dot{q}}{2k} L^2 - C_1 L + T_0 \qquad (4)$$

Subtract

$$0 = 0 + 2LC_1 + 0$$

$C_1 = 0$ $\qquad$ Sub. in Eq. (2)

$$T = -\frac{\dot{q}}{2k} x^2 + T_0$$

$$T - T_0 = -\frac{\dot{q}}{2k} x^2 \qquad (5)$$