CHAPTER FOUR

Two Dimensional, Steady State Conduction

For two dimensional, steady state conditions with no generation and constant thermal conductivity, the general conduction equation reduces to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (4.1)$$

Methods for solving Eq. (4.1) include the use of

1- Analytical method.
2- Graphical method.
3- Numerical method (finite difference, finite element, or boundary element).

The best alternative is often one that uses a numerical technique

For two-dimensional problems involve complicated geometries and/or boundary conditions, the best alternative is often one that uses a numerical technique.

4.1 Numerical method

4.1.1 Finite Difference Form of the Heat Equation

Consider a two dimensional body that is to be divided into equal increments in both the *x* and *y* directions, as shown in Figure (4.1). The nodal points are designated as shown, the (*m*) locations indicating the *x* increment and the (*n*) locations indicating the *y* increment. We wish to establish the temperatures at any of these nodal points within the body, using Eq. (4.1) as a governing condition. Finite differences are used to approximate differential increments in the temperature and space coordinates; and the smaller we choose these finite increments, the more closely the true temperature distribution will be approximated.

A numerical solution enables determination of the temperature at only discrete points. The first step in any numerical analysis must therefore be to select these points. Referring to Figure (4.1), this may be done by subdividing the medium of interest into a number of small regions and assigning to each a reference point that is at its center. The reference point is frequently termed a nodal point (or simply a node), and the aggregate of points is termed a nodal network, grid, or mesh. The nodal points are designated by a numbering scheme that, for a two dimensional system, may take the form shown in Figure (4.1 a). The *x* and *y* locations are designated by the *m* and *n* indices, respectively.

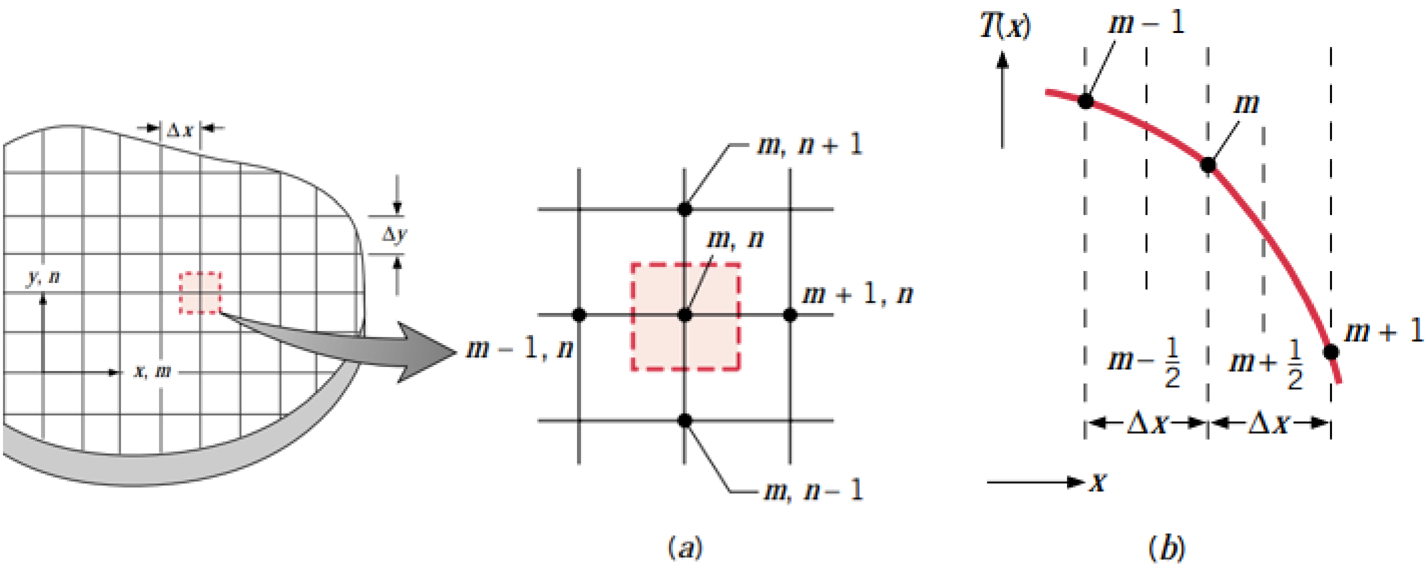


Figure (4.1) Two dimensional conduction. (a) Nodal network. (b) Finite difference approximation.

The temperature gradients may be written as follows:

$$\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} \approx \frac{T_{m+1,n} - T_{m,n}}{\Delta x} \qquad (4.2)$$

$$\left.\frac{\partial T}{\partial x}\right]_{m-1/2,n} \approx \frac{T_{m,n} - T_{m-1,n}}{\Delta x} \qquad (4.3)$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} \approx \frac{T_{m,n+1} - T_{m,n}}{\Delta y} \qquad (4.4)$$

$$\left.\frac{\partial T}{\partial y}\right]_{m,n-1/2} \approx \frac{T_{m,n} - T_{m,n-1}}{\Delta y} \qquad (4.5)$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial x}\right]_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right]_{m-1/2,n}}{\Delta x} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} \qquad (4.6)$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right]_{m,n} \approx \frac{\left.\frac{\partial T}{\partial y}\right]_{m,n+1/2} - \left.\frac{\partial T}{\partial y}\right]_{m,n-1/2}}{\Delta y} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} \qquad (4.7)$$

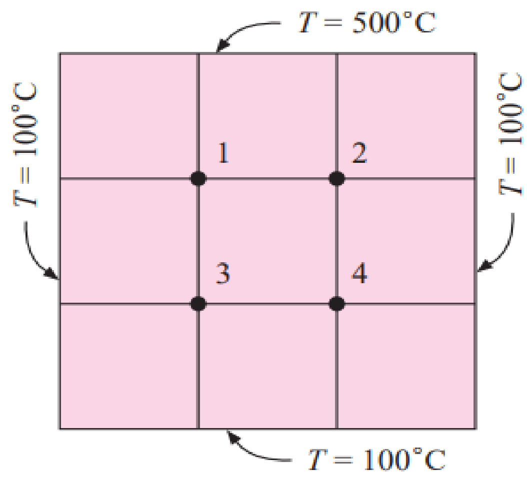
Substitute Eq's from Eq. (4.6) and Eq. (4.7) in to Eq. (4.1) get

$$\frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}=0$$

If $\Delta x = \Delta y$, then

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0 \qquad (4.8)$$

Example (4.1): Use the numerical method to write the finite difference equation for each node that shown in Figure



Solution:

$100 + T_2 + +500 + T_3 - 4T_1 = 0 \qquad (1)$

$T_1 + 100 + 500 + T_4 - 4T_2 = 0 \qquad (2)$

$100 + T_4 + T_1 + 100 - 4T_3 = 0 \qquad (3)$

$T_3 + 100 + T_2 + 100 - 4T_4 = 0 \qquad (4)$

4.1.2 The Energy Balance Method

In the energy balance method, the finite difference equation for a node is obtained by applying conservation of energy to a control volume about the nodal region.

Since the actual direction of heat flow (into or out of the node) is often unknown, it is convenient to formulate the energy balance by assuming that all the heat flow is into the node. Such a condition is, of course, impossible, but if the rate equations are expressed in a manner consistent with this assumption, the correct form of the finite difference equation is obtained. For steady state conditions with generation.

$$\dot{E}_{in}+\dot{E}_g=0 \qquad (4.9)$$

Consider applying Eq. (4.9) to a control volume about the interior node (*m*, *n*) of Figure (4.2). For two dimensional conditions, energy exchange is influenced by

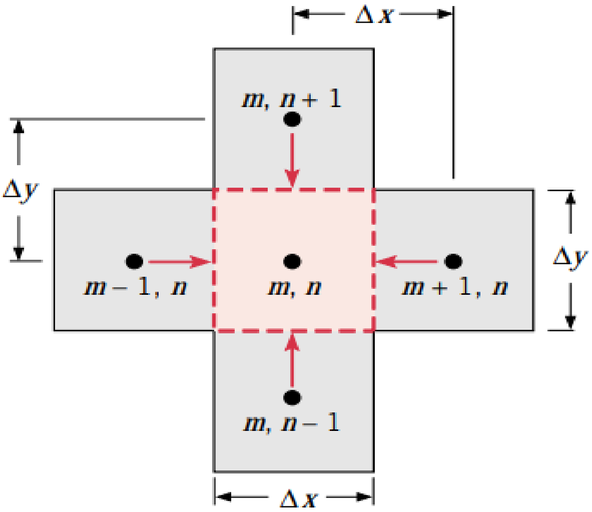


Figure (4.2) Conduction to an Interior Node from its Adjoining Nodes.

$$\sum_{i=1}^{4} q_{i\rightarrow(m,n)} + \dot{q}(\Delta x * \Delta y * 1) = 0 \qquad (4.10)$$

where *i* refers to the neighboring nodes, $q_{(i)\rightarrow(m,n)}$ is the conduction rate between nodes, and unit depth is assumed.

To evaluate the conduction rate terms, we assume that conduction transfer occurs exclusively through lanes that are oriented in either the *x* or *y* direction. Simplified forms of Fourier's law may therefore be used.

For example, the rate at which energy is transferred by conduction from node (*m*-1, *n*) to (*m*, *n*) may be expressed as

$$q_{(m-1,n)\rightarrow(m,n)} = k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} \qquad (4.11)$$

where ($\Delta y.1$) is the heat transfer area.

$(T_{m-1,n} - T_{m,n})/\,\Delta x$ is the finite difference approximation to the temperature gradient at the boundary between the two nodes.

The remaining conduction rates may be expressed as

$$q_{(m+1,n)\rightarrow(m,n)} = k(\Delta y \cdot 1)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} \qquad (4.12)$$

$$q_{(m,n+1)\rightarrow(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n+1} - T_{m,n}}{\Delta y} \qquad (4.13)$$

$$q_{(m,n-1)\rightarrow(m,n)} = k(\Delta x \cdot 1)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} \qquad (4.14)$$

Substitute Eq's from Eq. (4.11) to Eq. (4.14) into Eq. (4.10), so that the finite difference equation for an interior node with generation is

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0 \qquad (4.15)$$

$$\sum T_{neighbors} + \frac{\dot{q}(\Delta x)^2}{k} - 4T_{m,n} = 0 \qquad (4.16)$$

When the solid is exposed to some convection boundary condition, the temperatures at the surface must be computed differently from the method given above. Consider the boundary shown in Figure (4.3). The energy balance on node (m, n) is

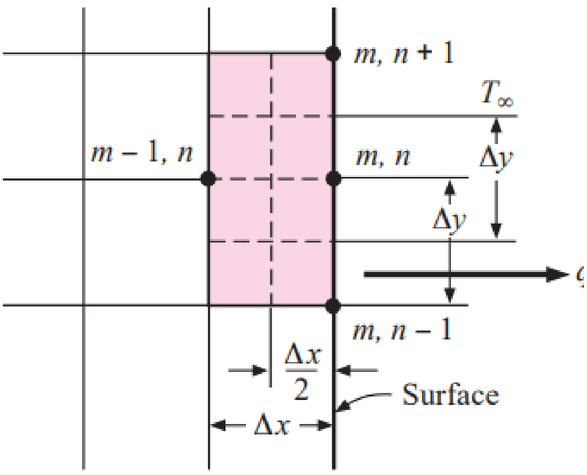


Figure (4.3) Nomenclature for Nodal Equation with Convective Boundary Condition.

$Q_{cond} = Q_{conv}$

$$k\Delta y\frac{(T_{m-1,n}-T_{m,n})}{\Delta x}+k\frac{\Delta x}{2}\frac{(T_{m,n+1}-T_{m,n})}{\Delta y}+k\frac{\Delta x}{2}\frac{(T_{m,n-1}-T_{m,n})}{\Delta y}=h\Delta y(T_{m,n}-T_\infty)$$

For $\Delta x = \Delta y$, and multiply by $(2/k)$ the boundary temperature is expressed in the equation

$$2(T_{m-1,n}-T_{m,n})+(T_{m,n+1}-T_{m,n})+(T_{m,n-1}-T_{m,n})=\frac{2h\Delta x}{k}(T_{m,n}-T_\infty)$$

$$2T_{m-1,n}-2T_{m,n}+T_{m,n+1}-T_{m,n}+T_{m,n-1}-T_{m,n}=\frac{2h\Delta x}{k}(T_{m,n}-T_\infty)$$

$$2T_{m-1,n}+T_{m,n+1}+T_{m,n-1}+(-2-1-1)T_{m,n}=\frac{2h\Delta x}{k}(T_{m,n}-T_\infty)$$

$$2T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}-\frac{2h\Delta x}{k}(T_{m,n}-T_\infty)=0$$

$$2T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}-\frac{2h\Delta x}{k}T_{m,n}+\frac{2h\Delta x}{k}T_\infty=0$$

$$(2T_{m-1,n}+T_{m,n+1}+T_{m,n-1})+\frac{2h\Delta x}{k}T_\infty-2\left(\frac{h\Delta x}{k}+2\right)T_{m,n}=0 \qquad (4.17)$$

Eq. (4.17) applies to a plane surface exposed to a convection boundary condition. It will not apply for other situations, such as an insulated wall or a corner exposed to a convection boundary condition.

Consider the corner section shown in Figure (4.4). The energy balance for the corner section is

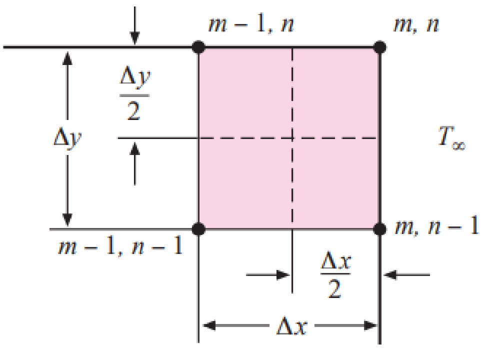


Figure (4.4) Nomenclature for Nodal Equation with Convection at a Corner Section

$Q_{cond} = Q_{conv}$

$$k\frac{\Delta y}{2}\frac{(T_{m-1,n} - T_{m,n})}{\Delta x} + k\frac{\Delta x}{2}\frac{(T_{m,n-1} - T_{m,n})}{\Delta y} = h\frac{\Delta x}{2}(T_{m,n} - T_{\infty}) + h\frac{\Delta y}{2}(T_{m,n} - T_{\infty})$$

For $\Delta x = \Delta y$ and multiply by $(2/k)$ the boundary temperature is expressed in the equation

$$T_{m-1,n} - T_{m,n} + T_{m,n-1} - T_{m,n} = \frac{h\Delta x}{k}(T_{m,n} - T_{\infty}) + \frac{h\Delta x}{k}(T_{m,n} - T_{\infty})$$

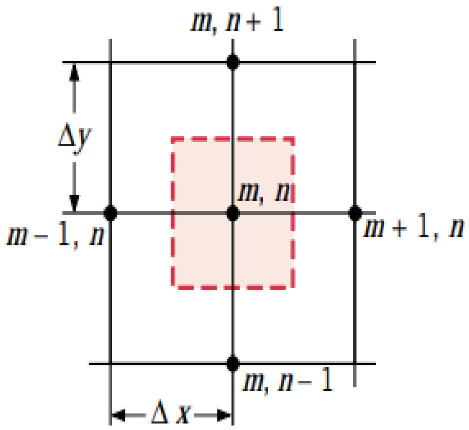
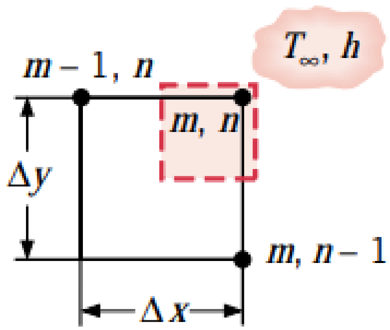
$$T_{m-1,n} + T_{m,n-1} - 2T_{m,n} = 2\frac{h\Delta x}{k}(T_{m,n} - T_{\infty})$$

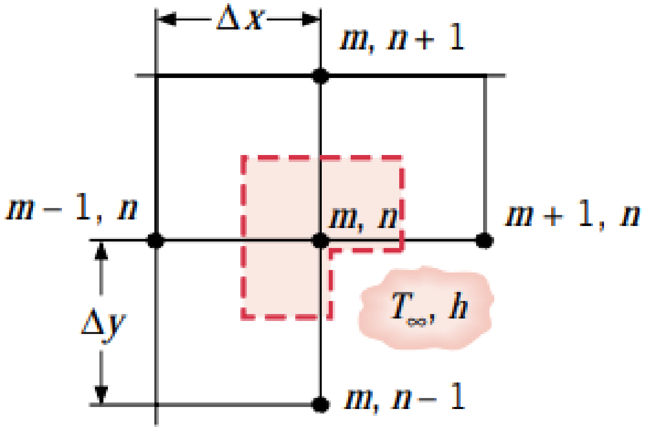
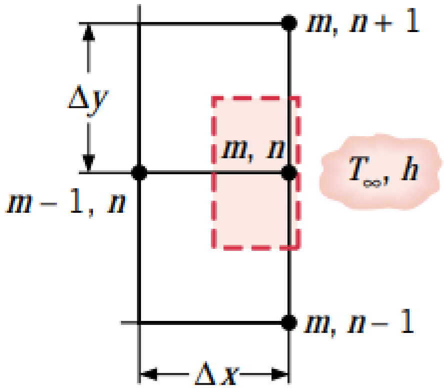
$$T_{m-1,n} + T_{m,n-1} - 2T_{m,n} - 2\frac{h\Delta x}{k}(T_{m,n} - T_{\infty}) = 0$$

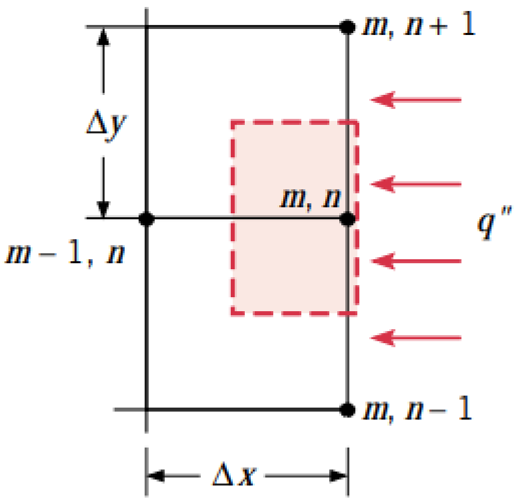
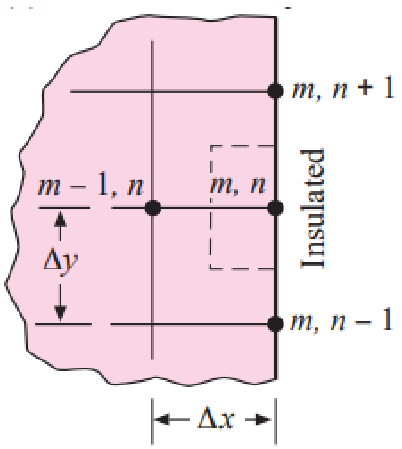
$$T_{m-1,n} + T_{m,n-1} - 2T_{m,n} - 2\frac{h\Delta x}{k}T_{m,n} + 2\frac{h\Delta x}{k}T_{\infty} = 0$$

$$T_{m-1,n} + T_{m,n-1} + 2\frac{h\Delta x}{k}T_{\infty} - (2\frac{h\Delta x}{k} + 2)T_{m,n} = 0$$

$$(T_{m,n-1} + T_{m-1,n}) + 2\frac{h\,\Delta x}{k}T_{\infty} - 2\left(\frac{h\,\Delta x}{k} + 1\right)T_{m,n} = 0 \qquad (4.18)$$

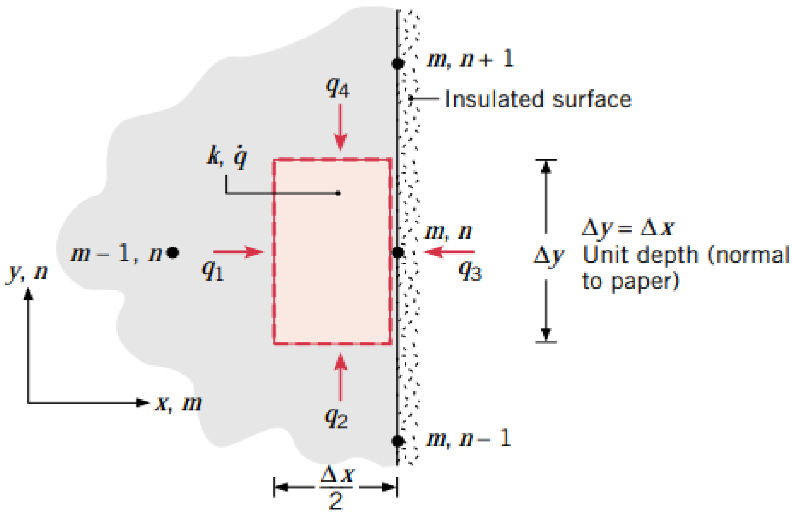
Configuration	Finite Difference Equation for $\Delta x = \Delta y$
	CASE (1) Interior node $$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0$$
	CASE (2) Node at an external corner with convection $$(T_{m,n-1} + T_{m-1,n}) + 2\frac{h\,\Delta x}{k}T_\infty - 2\left(\frac{h\,\Delta x}{k} + 1\right)T_{m,n} = 0$$

	CASE (3) Node at an internal corner with convection $T_{m-1,n} + T_{m,n+1} + \frac{1}{2}(T_{m+1,n} + T_{m,n-1}) + \frac{h\Delta x}{k} T_\infty - \left(3 + \frac{h\Delta x}{k}\right) T_{m,n} = 0$
	CASE (4) Node at a plane surface with convection $(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2h\Delta x}{k} T_\infty - 2\left(\frac{h\Delta x}{k} + 2\right) T_{m,n} = 0$ **Note:** To obtain the finite difference equation for an adiabatic surface (or surface of symmetry), simply set (*h*) equal to zero.

	CASE (5) Node at a plane surface with uniform heat flux $$(2T_{m-1,n} + T_{m,n+1} + T_{m,n-1}) + \frac{2q'' \Delta x}{k} - 4T_{m,n} = 0$$ **Note:** To obtain the finite difference equation for an adiabatic surface (or surface of symmetry), simply set (q'') equal to zero.
	CASE (6) Insulated boundary $$T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n} - 4T_{m,n} = 0$$

Example (4.2): Using the energy balance method, derive the finite difference equation for the (*m*, *n*) nodal point located on a plane, insulated surface of a medium with uniform heat generation.

Solution:



Assumptions:

1. Steady-state conditions.

2. Two dimensional conduction.

3. Constant properties.

4. Uniform internal heat generation.

$$q_1 + q_2 + q_3 + q_4 + \dot{q}\left(\frac{\Delta x}{2}\cdot \Delta y \cdot 1\right) = 0 \qquad (1)$$

where

$$q_1 = k(\Delta y \cdot 1)\frac{T_{m-1,n} - T_{m,n}}{\Delta x}$$

$$q_2 = k\left(\frac{\Delta x}{2}\cdot 1\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y}$$

$$q_3 = 0$$

$$q_4 = k\left(\frac{\Delta x}{2}\cdot 1\right)\frac{T_{m,n+1} - T_{m,n}}{\Delta y}$$

Substituting into Eq. (1) and dividing by *k*/2, it follows that

$$2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} + \frac{\dot{q}(\Delta x \cdot \Delta y)}{k} = 0 \qquad \text{for } \Delta x = \Delta y$$

$$2T_{m-1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} + \frac{\dot{q}\Delta x^2}{k} = 0$$

4.2 Solving the Finite Difference Equations

As examples of the direct and iterative methods to solve the finite difference equation is

1- Matrix inversion.
2- Gauss Seidel iteration.

4.2.1 The Matrix Inversion Method

Consider a system of *N* finite-difference equations corresponding to (*N*) unknown temperatures. Identifying the nodes by a single integer subscript, rather than by the double subscript (*m*, *n*), the procedure for performing a matrix inversion begins by expressing the equations as

$$\begin{array}{l} a_{11}T_1 + a_{12}T_2 + a_{13}T_3 + \cdots + a_{1N}T_N = C_1 \\ a_{21}T_1 + a_{22}T_2 + a_{23}T_3 + \cdots + a_{2N}T_N = C_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{N1}T_1 + a_{N2}T_2 + a_{N3}T_3 + \cdots + a_{NN}T_N = C_N \end{array}$$

where the quantities $a_{11}, a_{12}, \ldots, C_1, \ldots$ are known coefficients and constants involving quantities such as $\Delta x, k, h, and\ T_\infty$. Using matrix notation, these equations may be expressed as

$$[A][T] = [C] \qquad (4.19)$$

where

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, \quad T \equiv \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, \quad C \equiv \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$

The solution of Eq. (4.19) be expressed as

$$[T] = [A]^{-1}[C] \tag{4.20}$$

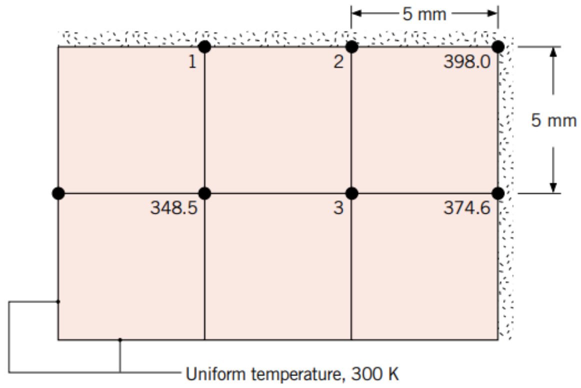
where $[A]^{-1}$ is the inverse of $[A]$ and is defined as

$$[A]^{-1} \equiv \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \vdots & \vdots & & \vdots \\ b_{N1} & b_{N2} & \cdots & b_{NN} \end{bmatrix}$$

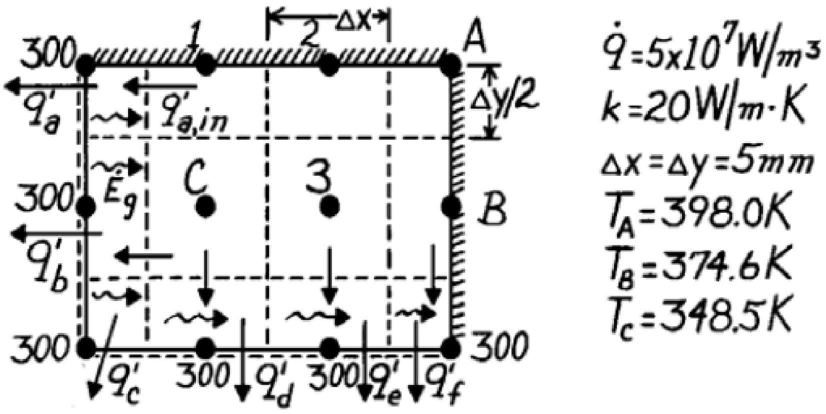
Evaluating the right hand side of Eq. (4.20), it follows that

$$\begin{aligned} T_1 &= b_{11}C_1 + b_{12}C_2 + \cdots + b_{1N}C_N \\ T_2 &= b_{21}C_1 + b_{22}C_2 + \cdots + b_{2N}C_N \\ &\vdots \\ T_N &= b_{N1}C_1 + b_{N2}C_2 + \cdots + b_{NN}C_N \end{aligned}$$

Example (4.3): Steady state temperatures (K) at three nodal points of a long rectangular rod are as shown. The rod experiences a uniform volumetric generation rate of $(5 \times 10^7\, W/m^3)$ and has a thermal conductivity of (20 *W/m. K*). Two of its sides are maintained at a constant temperature of (300 K), while the others are insulated. Determine the temperatures at nodes 1, 2, and 3.



Solution:



Node (A) to find T_2

Node at an external corner without convection (i.e *h*=0) and with heat generation (i. e added the term $(\dot{q}V/k)$ to the equation)

$$T_{m-1,n} + T_{m,n-1} - 2T_{m,n} + \frac{\dot{q}V}{k} = 0$$

$$\Delta x = \Delta y$$

$$V = \frac{\Delta x}{2} * \frac{\Delta y}{2} * 1 = \frac{\Delta x^2}{4}$$

$$T_{m-1,n} + T_{m,n-1} - 2T_{m,n} + \frac{\dot{q}\Delta x^2}{4k} = 0$$

$$T_2 + T_B - 2T_A + \frac{\dot{q}\Delta x^2}{4k} = 0$$

$$T_2 = 2T_A - T_B - \frac{\dot{q}\Delta x^2}{4k}$$

$$T_2 = 2 * 398 - 374.6 - \frac{5 \times 10^7 * (0.005)^2}{4 * 20}$$

$$T_2 = 405.77\ K$$

Node (3) to find T_3

Interior node with heat generation (i. e added the term ($\dot{q}V/k$) to the equation)

$$T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} + \frac{\dot{q}V}{k} = 0$$

$$\Delta x = \Delta y$$

$$V = \Delta x * \Delta y * 1 = \Delta x^2$$

$$T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1} - 4T_{m,n} + \frac{\dot{q}\Delta x^2}{k} = 0$$

$$T_C + T_B + 300 + T_2 - 4T_3 + \frac{\dot{q}\Delta x^2}{k} = 0$$

$$T_3 = \frac{1}{4}(T_C + T_B + 300 + T_2 + \frac{\dot{q}\Delta x^2}{k})$$

$$T_3 = \frac{1}{4}(348.5 + 374.6 + 300 + 405.77 + \frac{5 \times 10^7 * (0.005)^2}{20})$$

$$T_3 = 372.84\ K$$

Node (1) to find T_1

Insulated boundary with heat generation (i. e added the term ($\dot{q}V/k$) to the equation)

$$T_{m-1,n} + T_{m+1,n} + 2T_{m,n-1} - 4T_{m,n} + \frac{\dot{q}V}{k} = 0$$

$$\Delta x = \Delta y$$

$$V = \Delta x * \frac{\Delta y}{2} * 1 = \frac{\Delta x^2}{2}$$

$$T_{m-1,n} + T_{m+1,n} + 2T_{m,n-1} - 4T_{m,n} + \frac{\dot{q}\Delta x^2}{2k} = 0$$

$$300 + T_2 + 2T_C - 4T_1 + \frac{\dot{q}\Delta x^2}{2k} = 0$$

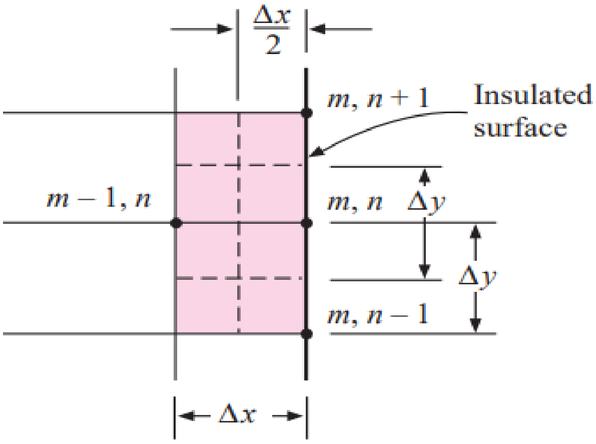
$$T_1 = \frac{1}{4}(300 + T_2 + 2T_C + \frac{\dot{q}\Delta x^2}{2k})$$

$$T_1 = \frac{1}{4}(300 + 405.77 + 2 * 348.5 + \frac{5 \times 10^7 * (0.005)^2}{2 * 20})$$

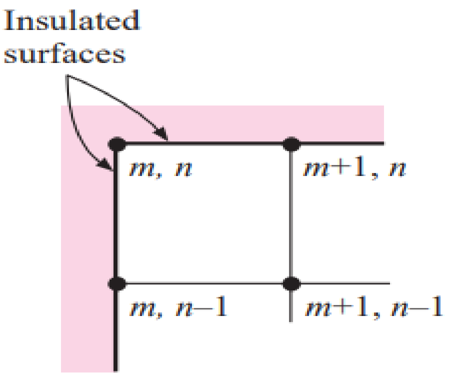
$$T_1 = 358.5\ K$$

Home Work (4):

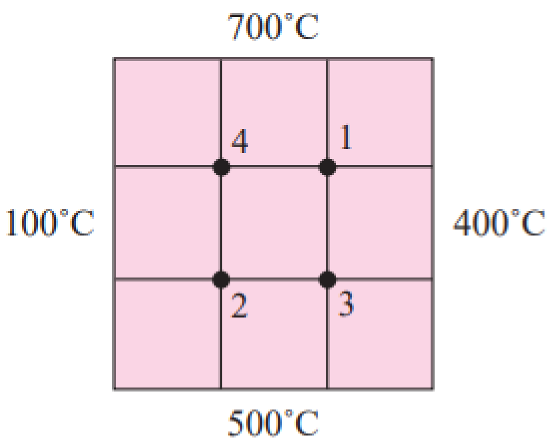
1- Show that the nodal equation corresponding to an insulated wall shown in Figure below is $T_{m,n+1} + T_{m,n-1} + 2T_{m-1,n} - 4T_{m,n} = 0$



2- For the insulated corner section shown in Figure below, derive an expression for the nodal equation of node (m, n) under steady-state conditions.



3- In Figure below calculate the temperatures at points 1, 2 and 3 take $T_4 = 413$ ℃ using the numerical method.

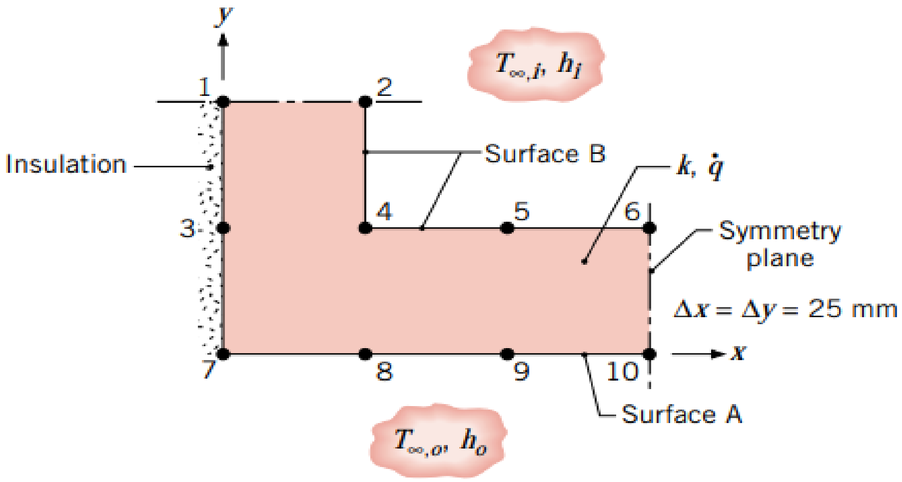


4- Steady state temperatures at selected nodal points of the symmetrical section of a flow channel are known to be (T_2 = 95.47 °C), (T_3 = 117.3 °C), (T_5 = 79.79 °C , (T_6 = 77.29 °C), (T_8 = 87.28 °C), *and* (T_{10} = 77.65 °C) . The wall experiences uniform volumetric heat generation of ($10^6\ W/m^3$) and has a thermal conductivity of ($k = 10\ W/m.K$). The inner and outer surfaces of the channel experience convection with fluid temperatures of ($T_{\infty,i}$ = 50 °C) and ($T_{\infty,o}$ = 25 °C) and convection coefficients of ($h_i = 500\ W/m^2.K$) and ($h_0 = 250\ W/m^2.K$).

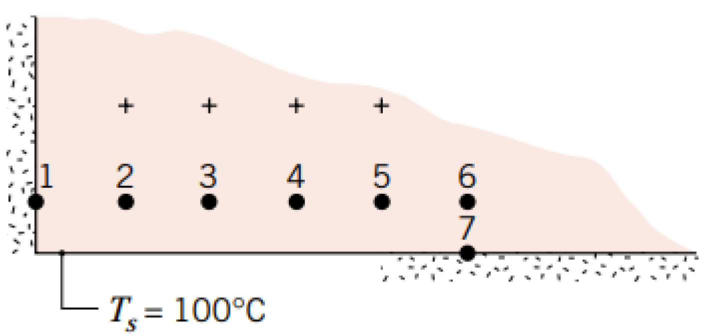
(a) Determine the temperatures at nodes 1, 4, 7, and 9.

(b) Calculate the heat rate per unit length (W/m) from the outer surface A to the adjacent fluid.

(c) Calculate the heat rate per unit length from the inner fluid to surface B.



5- Consider the network for a two-dimensional system without internal volumetric generation having nodal temperatures shown below. If the grid space is (125 mm) and the thermal conductivity of the material is (50 W/m. K), calculate the heat rate per unit length normal to the page from the isothermal surface (T_s).



Node	T_i (°C)
1	120.55
2	120.64
3	121.29
4	123.89
5	134.57
6	150.49
7	147.14