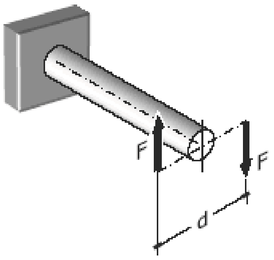
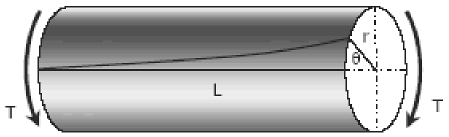
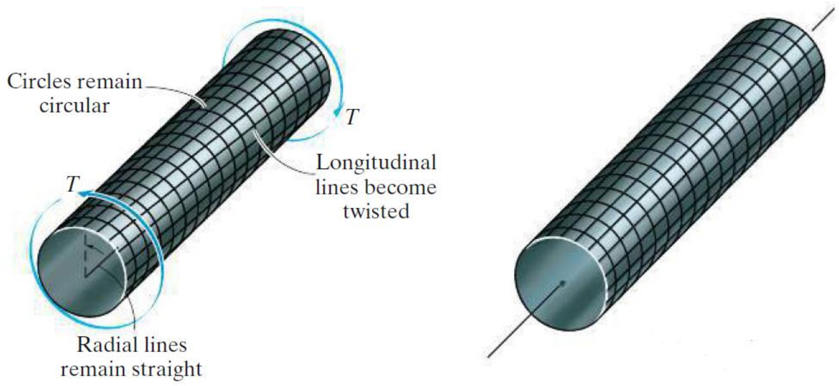
Torsion

Consider a bar to be rigidly attached at one end and twisted at the other end by a torque or twisting moment T equivalent to F × d, which is applied perpendicular to the axis of the bar, as shown in the figure. Such a bar is said to be in torsion.

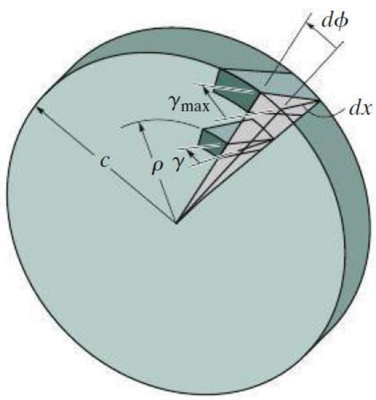






Torsional Shearing Stress, τ

For a solid or hollow circular shaft subject to a twisting moment T, the torsional shearing stress τ at a distance ρ from the center of the shaft is



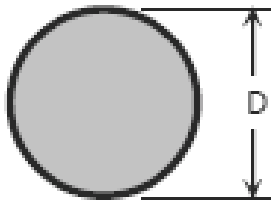
$$\tau = \frac{T\rho}{J} \text{ and } \tau_{max} = \frac{Tr}{J}$$

where J is the polar moment of inertia of the section and r is the outer radius.

- For solid cylindrical shaft:

$$J = \frac{\pi}{32} D^4$$

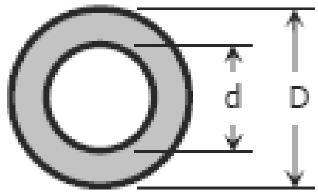
$$\tau_{max} = \frac{16T}{\pi D^3}$$



- For hollow cylindrical shaft:

$$J = \frac{\pi}{32}(D^4 - d^4)$$

$$\tau_{max} = \frac{16TD}{\pi(D^4 - d^4)}$$



Angle of Twist

The angle θ through which the bar length L will twist is

$$\theta = \frac{TL}{JG} \text{ in radians}$$

where T is the torque in N·mm, L is the length of shaft in mm, G is shear modulus in MPa, J is the polar moment of inertia in mm4, D and d are diameter in mm, and r is the radius in mm.

Power Transmitted by The Shaft

A shaft rotating with a constant angular velocity ω (in radians per second) is being acted by a twisting moment T. The power transmitted by the shaft is

$$P = T\omega = 2\pi T f$$

where T is the torque in N·m, f is the number of revolutions per second, and P is the power in watts.

Problem 1: A steel shaft 3 ft long that has a diameter of 4 in. is subjected to a torque of 15 kip·ft. Determine the maximum shearing stress and the angle of twist. Use G = 12 × 106 psi.

Solution

$$\tau_{max} = \frac{16T}{\pi D^3} = \frac{16(15)(1000)(12)}{\pi(4^3)}$$

$$\tau_{max} = 14\ 324 \text{ psi}$$

$$\tau_{max} = 14.3 \text{ ksi}$$

$$\theta = \frac{TL}{JG} = \frac{15(3)(1000)(12^2)}{\frac{1}{32}\pi(4^4)(12\times10^6)}$$

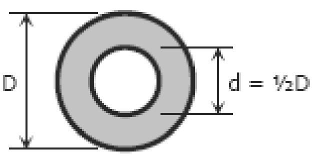
$$\theta = 0.0215 \text{ rad}$$

$$\theta = 1.23°$$

Problem 2: Show that the hollow circular shaft whose inner diameter is half the outer diameter has a torsional strength equal to 15/16 of that of a solid shaft of the same outside diameter.

Solution

Hollow circular shaft:



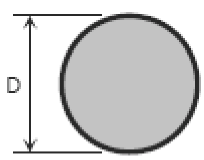
$$\tau_{max\text{-}hollow} = \frac{16TD}{\pi(D^4 - d^4)}$$

$$= \frac{16TD}{\pi[D^4 - (\frac{1}{2}D)^4]}$$

$$= \frac{16TD}{\pi(\frac{15}{16}D^4)}$$

$$= \frac{16^2 T}{15\pi D^3}$$

Solid circular shaft:

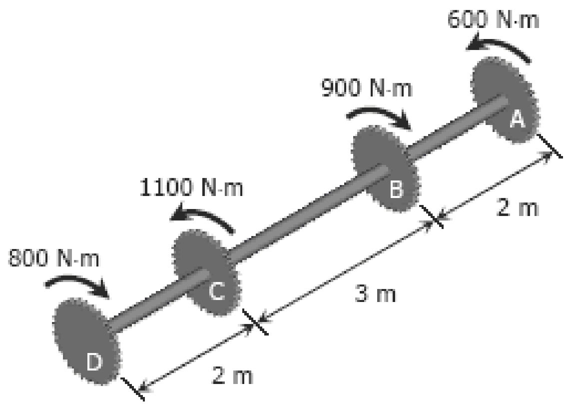


$$\tau_{max\text{-}solid} = \frac{16T}{\pi D^3}$$

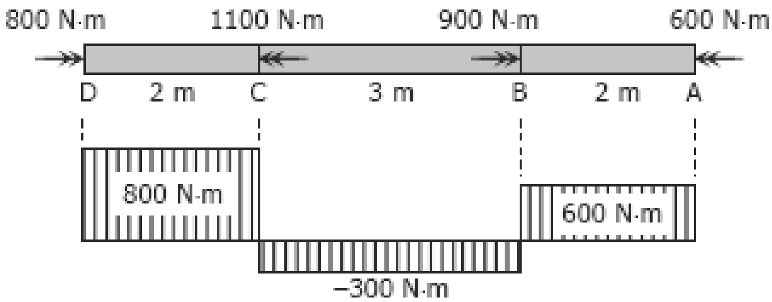
$$= \frac{15}{16}\left[\frac{16^2 T}{15\pi D^3}\right]$$

$$= \frac{15}{16} \times \tau_{max\text{-}hollow} \quad \textit{ok!}$$

Problem 3: An aluminum shaft with a constant diameter of 50 mm is loaded by torques applied to gears attached to it as shown in Fig. Using G = 28 GPa, determine the relative angle of twist of gear D relative to gear A.



Solution



$$\theta = \frac{TL}{JG}$$

Rotation of D relative to A:

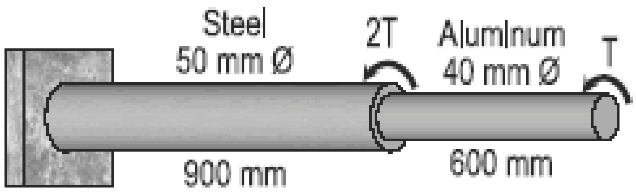
$$\theta_{D/A} = \frac{1}{JG}\sum TL$$

$$\theta_{D/A} = \frac{1}{\frac{1}{32}\pi(50^4)(28000)}\,[800(2) - 300(3) + 600(2)]\,(100^2)$$

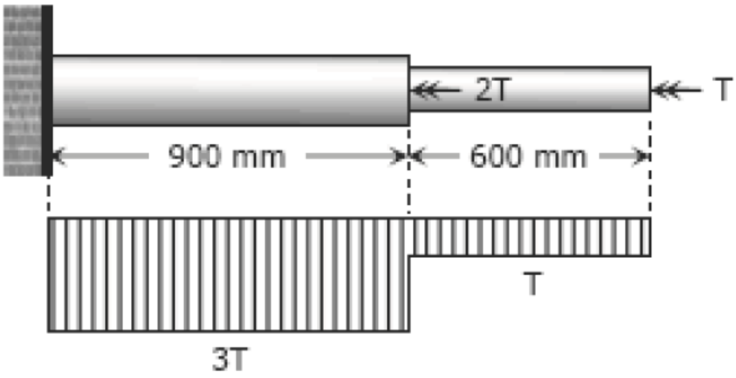
$$\theta_{D/A} = 0.1106 \text{ rad}$$

$$\theta_{D/A} = 6.34°$$

Problem 4: A compound shaft consisting of a steel segment and an aluminum segment is acted upon by two torques as shown in Fig. Determine the maximum permissible value of T subject to the following conditions: τ_{st} = 83 MPa, τ_{al} = 55 MPa, and the angle of rotation of the free end is limited to 6°. For steel, G = 83 GPa and for aluminum, G = 28 GPa.



Solution



Based on maximum shearing stress $\tau_{max} = 16T / \pi d^3$:

$$\tau_{st} = \frac{16(3T)}{\pi(50^3)} = 83$$

$T = 679\,042.16$ N·mm

$T = 679.04$ N·m

$$\tau_{al} = \frac{16T}{\pi(40^3)} = 55$$

$T = 691\,150.38$ N·mm

$T = 691.15$ N·m

Based on maximum angle of twist:

$$\theta = \left(\frac{TL}{JG}\right)_{st} + \left(\frac{TL}{JG}\right)_{al}$$

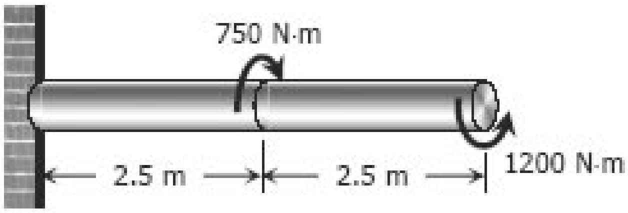
$$6°\left(\frac{\pi}{180°}\right) = \frac{3T(900)}{\frac{1}{32}\pi(50^4)(83000)} + \frac{T(600)}{\frac{1}{32}\pi(40^4)(28000)}$$

$T = 757\,316.32$ N·mm

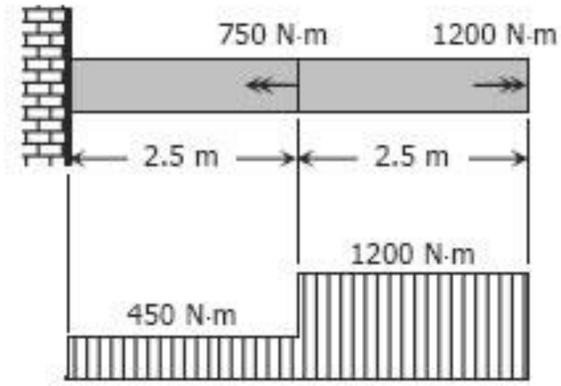
$T = 757.32$ N·m

Use T = **679.04 N·m**

Problem 5: A solid steel shaft is loaded as shown in Fig. Using G = 83 GPa, determine the required diameter of the shaft if the shearing stress is limited to 60 MPa and the angle of rotation at the free end is not to exceed 4 deg.



Solution



Based on maximum allowable shear:

$$\tau_{max} = \frac{16T}{\pi D^3}$$

For the 1st segment:

$$60 = \frac{450(2.5)(1000^2)}{\pi D^3}$$

$$D = 181.39 \text{ mm}$$

For the 2nd segment:

$$60 = \frac{1200(2.5)(1000^2)}{\pi D^3}$$

$$D = 251.54 \text{ mm}$$

Based on maximum angle of twist:

$$\theta = \frac{TL}{JG}$$

$$\theta = \frac{1}{JG}\sum TL$$

$$4°\left(\frac{\pi}{180°}\right) = \frac{1}{\frac{1}{32}\pi D^4(83000)}[450(2.5) + 1200(2.5)]\,(1000^2)$$

$$D = 51.89 \text{ mm}$$

Use D = **251.54 mm**