Al-Mustaqbal University College

Department of Medical Instrumentation Techniques Engineering

LECTURE TWO

Derivative and Application of derivative

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Lecture Two: The Derivative

Definition: The derivative of the function $y = f(x)$ with respect to the variable x is the function y' or $\acute{f}(x)$whose value at x is

$$\acute{f}(x_0) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Provided this limit exists.

Example: Use definition to find $\frac{dy}{dx}$ if $y = f(x) = x^3$

Sol:

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \longrightarrow \lim_{h\to 0}\frac{f(x+h)^3-x^3}{h}$$

$$= \lim_{h\to 0}\frac{x^3+3x^2h+3h^2x+h^3-x^3}{h} \longrightarrow \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2$$

Differentiation Rules:

1-Derivative of a constant function.

If $y = f(x) = c$ *where c is constant then,*

$$\frac{dy}{dx} = \acute{f}(x) = 0$$

Example: $f(x) = 1$ *then* $\frac{dy}{dx} = \acute{f}(x) = 0$

2-Power rule for positive integers:

If n is a positive integer, and $y = f(x) = x^n$;then

$$\acute{f}(x) = \frac{dy}{dx} = n\, x^{n-1}$$

Example: $y = f(x) = x^3$

Sol: $\acute{f}(x) = \frac{dy}{dx} = 3\, x^2$

3-Derivative constant multiple rule:

If u is a differentiable function of x $f(x) = c\, u(x)$, and c is a constant, then

$$\frac{dy}{dx} = \acute{f}(x) = c\acute{u}(x)$$

Example: $f(x) = 5x^6$ $then\ \frac{dy}{dx} = 6 * 5\, x = 30x^5$

4-Derivative Product rule:

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example: Find the derivative of $y = (x^2 + 8x)(x^3 - 1)$

Sol:

$$\frac{dy}{dx} = (x^2 + 8x).3x^2 + (x^3 - 1).(2x + 8)$$

5-Derivative quotient rule:

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient $\frac{u}{v}$ is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Example: Find the derivative of $y=\frac{t-t^2}{t+4}$

Sol:

$$\frac{d}{dt}\left(\frac{t-t^2}{t+4}\right)=\frac{(t+4).(1-2t)-(t-t^2).1}{(t+4)^2}=\frac{(t+4)(1-2t)-(t-t^2)]}{(t+4)^2}$$

Derivative Rules:

General formulas		Trigonometric functions
(1) $\frac{d}{dx}(c)=0$ (The derivative of a constant function is zero.)		(1) $\frac{d}{dx}(\sin x)=\cos x$
(2) $\frac{d}{dx}(x)=1$ (The derivative of the identity function is 1.)		(2) $\frac{d}{dx}(\cos x)=-\sin x$
(3) $\frac{d}{dx}(cu)=c\frac{du}{dx}$	(Constant Multiple)	(3) $\frac{d}{dx}(\tan x)=\sec^2 x$
(4) $\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$	(Sum Rule)	(4) $\frac{d}{dx}(\cot x)=-\csc^2 x$
(5) $\frac{d}{dx}(u-v)=\frac{du}{dx}-\frac{dv}{dx}$	(Difference Rule)	(5) $\frac{d}{dx}(\sec x)=\sec x\tan x$
(6) $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	(Product Rule)	6) $\frac{d}{dx}(\csc x)=-\csc x\cot x$
(7) $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ provided that $v\neq 0$	(Quotient Rule)	**Chain rule** Let $u=g(x)$ and $y=f(u)$. Then, $y=f(u)=f(g(x))$ $y'=f'(g(x))*g'(x)$ or $\frac{dy}{dx}=\frac{dy}{du}*\frac{du}{dg}$
(8) $\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{-1}{x^2}$	provided that $x\neq 0$	Where $\frac{dy}{dx}$ is evaluated
(9) $\frac{d}{dx}(x^m)=mx^{m-1}$	(Power Rule)	at $u=g(x)$

Example:

$$y=x^3+\frac{4}{3}x^2-5x+1 \rightarrow \frac{dy}{dx}=\frac{d}{dx}(x^3)+\frac{d}{dx}\left(\frac{4}{3}x^2\right)-\frac{d}{dx}(5x)+\frac{d}{dx}(1)$$

$$=3x^2+\frac{4}{3}*2x-5+0=3x^2+\frac{8}{3}x-5$$

Example:

$y = \frac{x^2-1}{x^2+1}$ we apply the quotient with $u = x^2 - 1$ and $v = x^2 + 1$

$$\frac{dy}{dx} = \frac{(x^2+1)*2x - (x^2-1)*2x}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$$
$$= \frac{4x}{(x^2+1)^2}$$

Example

$$y = (3x^2+1)^2$$
$$= 2(3x^2+1)^{2-1}.(3*2x^{2-1}+0) = 2(3x^2+1)*6x$$
$$= 36x^3 + 12x$$

Example: Find $\frac{dy}{dx}$ of the following functions $y = \sin(x^2+2x-5)$

Sol:

$$\frac{dy}{dx} = \cos(x^2+2x-5).2x+2 = (2x+2)\cos(x^2+2x-5)$$

Example $y = tan\,(2x)\cos(x^2+1)$

Sol:

$$\frac{dy}{dx} = -\tan(2x)\sin(x^2+1).2x + \cos(x^2+1)\sec^2(2x).2$$

$$\frac{dy}{dx} = -2x\tan(2x)\sin(x^2+1).+2\cos(x^2+1)\sec^2(2x)$$

Second and Higher- Order Derivatives:

If $f(x)$ is a given function then

$$\frac{dy}{dx} = \acute{f}(x)\ \ is\ \ first\ derivaive\ of\ y\,.$$

$\frac{d^2y}{dx^2} = f''(x)$ *is second derivaive of y .*

$\frac{d^3y}{dx^3} = f'''(x)$ *is third derivative of y*. And so on…

Then, in general: $\boldsymbol{\frac{d^n y}{dx^n} = f^n(x) = y^n}$

Higher Derivatives:	**Distance , Velocity and Acceleration :**
If y =f(x) ,then	Time : - t
First derivative: y', $f'(x)$, $\frac{dy}{dx}$	Distance - s(t)
Second derivative: y'', $f''(x)$, $\frac{d^2y}{dx^2}$	Velocity - v(t) , $\frac{ds}{dt}$
Third derivative: y''', $f'''(x)$, $\frac{d^3y}{dx^3}$	Acceleration - a(t) , $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$
nth derivative: y^n, $f^n(x)$, $\frac{d^n y}{dx^n}$	

Example: If $y = (x^2 + 2x + 3)^2$, *find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$

Sol:

$$\frac{dy}{dx} = \acute{y} = 2\,.(x^2 + 2x + 3)(2x + 2)$$

$$\frac{d^2y}{dx^2} = 2\,[(x^2 + 2x + 3).2 + (2x + 2).(2x + 2)]$$

$$\frac{d^2y}{dx^2} = 4(x^2 + 2x + 3) + 2(2x + 2)^2 .$$

Example: A body moves along a straight line according to the law $s = \frac{1}{2}t^3 - 2t$. Determine its velocity and acceleration at the end of 2 seconds.

Solution

$v = \frac{ds}{dt} = \frac{3}{2}t^2 - 2 = \frac{3}{2}2^2 - 2 = 4\ {}^{m}/{}_{s}$, $a = \frac{dv}{dt} = 3t = 3*2 = 6\ {}^{m}/{}_{s}{}^{2}$

Chain rule:

If y is a function of x , say $y = f(x)$, and x is a function of t, say $x = g(t)$ then y is a function of t:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

This formula is called chain rule.

Example: If $y = x^3 - x^2 + 5$ *and* $x = 2t^2 + t$, *find* $\frac{dy}{dt}$ *at* $t = 1$.

Sol:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3x^2 - 2x)(4t + 1)$$

$$at\ t = 1 \rightarrow x = (2)1^2 + 1 = 3$$

$$\frac{dy}{dt} = (3 * 3^2 - 2 * 3)(4 * 1 + 1) = 105$$

Example

Example: if $y = u^3$-1 and $u = 2x$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx} = 3u^2 * 2 = 6(2x)^2 = 24x^2$$

Implicit Differentiation:

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way.

Example: Find $\frac{dy}{dx}$ for the equation $y^2 + x^3 - 9xy = 0$

Sol:

$$2y\frac{dy}{dx} + 3x^2 - \left(9x\frac{dy}{dx} + 9y\right) = 0 \rightarrow 2y\frac{dy}{dx} + 3x^2 - 9x\frac{dy}{dx} - 9y = 0$$

$$2y\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2 \rightarrow \frac{dy}{dx}(2y - 9x) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y-3x^2}{2y-9x}$$

Example: $y^2 = \frac{x-1}{x+1}$

Sol:

$$2y\frac{dy}{dx} = \frac{(x+1).1 - (x-1).1}{(x-1)^2} = \frac{x+1-x+1}{(x-1)^2} = \frac{2}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{2}{2y(x-1)^2} = \frac{1}{y(x-1)^2}$$

Application of Differentiation

Indeterminate Forms and Hôpital's Rule:

Suppose that $f(a) = g(a) = 0$, that $\acute{f}(a), \acute{g}(a)$ exist, and that $\acute{g}(a) \neq 0$

Then;

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \frac{\acute{f}(a)}{\acute{g}(a)}$$

Example: Using Hospital's Rule and find the following:

1) $\lim_{x \to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1}\Big|_{x=0} = 2$

2) $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \frac{\frac{1}{2\sqrt{x+1}}}{1}\Bigg|_{x=0} = \frac{1}{2}$

3) $\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \frac{1}{2x}\Big|_{x=2} = \frac{1}{4}$

4) $\lim_{x \to 1} \frac{x^3 - 1}{4x^3 - x - 3} = \frac{3x^2}{12x^2 - 1}\Big|_{x=1} = \frac{3}{11}$

H.W

1. If $y = 2x^3 - 4x^2 + 6x - 5$, $find\ \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$

2. Use implicit differentiation to find dy/dx for $x\cos(2x+3)y \sin x$

3. Use implicit differentiation $x^2 + y^2 - 25 = 0$

4. find $\frac{dy}{dx}$ $y = 6u - 9$, $u = x^4/2$

5. $y = \sin^2\left(x^2 + \frac{1}{x^2}\right)$

6. If $y = \tan^{-3}(\sin 2x)$, $find\ \frac{dy}{dx}$