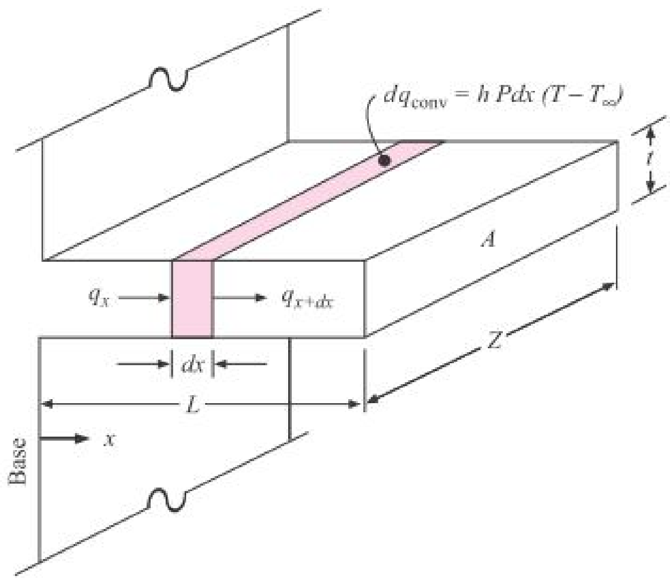
Conduction-Convection Systems

In heat-exchanger applications a finned-tube arrangement might be used to remove heat from a hot liquid. The heat transfer from the liquid to the finned tube is by convection. The extended-surface problem is a simple application of a conduction-convection systems. Consider the one-dimensional fin exposed to a surrounding fluid at a temperature $T\infty$ as shown in Figure below. The temperature of the base of the fin is T_0.

Let P = perimeter = 2(Z + t)



The energy balance on an element of the fin of thickness dx is

Energy in left face = energy out right face + energy lost by convection

$$\text{Energy in left face} = q_x = -kA\frac{dT}{dx}$$

$$\text{Energy out right face} = q_{x+dx} = -kA\left.\frac{dT}{dx}\right]_{x+dx}$$

$$= -kA\left(\frac{dT}{dx} + \frac{d^2T}{dx^2}dx\right)$$

$$\text{Energy lost by convection} = hP\,dx\,(T - T_\infty)$$

$$\therefore \quad -kA\frac{dT}{dx} = -kA\left[\frac{dT}{dx} + \frac{d^2T}{dx^2}dx\right] + hPdx(T - T_\infty), \qquad \div dx \Rightarrow$$

$$\frac{d^2T}{dx^2} - \frac{hP}{kA}(T - T_\infty) = 0$$

$$let \quad \theta = T - T_\infty, \quad \frac{dT}{dx} = \frac{d\theta}{dx} \quad \Rightarrow$$

$$\frac{d^2T}{dx^2} = \frac{d^2\theta}{dx^2}$$

Let $m^2 = hP/kA$

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0$$

(*D-Operator*)

$$D^2 - 1 = 0 \quad \Rightarrow \quad D = \pm 1$$

$\therefore$ The roots of the equation above are:

$$m_1 = +1, \quad m_2 = -1$$

The general solution is

$$\theta = C_1 e^{-mx} + C_2 e^{mx}$$

Several cases may be considered:

CASE 1: The fin is very long.

B.C.1 at $x = 0$ $\quad T = T_0 \quad \theta = \theta_0$

B.C.2 at $x = \infty$ $\quad T = T_\infty \quad \theta = 0$

By applying the boundary conditions on the general equation above, we get

$$\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_0 - T_\infty} = e^{-mx}$$

CASE 2: The end of the fin is insulated so that $dT/dx = 0$ at $x = L$.

B.C.1 at $x = 0$ $\quad T = T_0 \quad \theta = \theta_0$

B.C.2 at $x = L$ $\quad dT/dx = 0 \quad d\theta/dx = 0$

From B.C.1 $\quad \theta_0 = C_1 + C_2 \quad \Rightarrow C_1 = \theta_0 - C_2$

From B.C.2 $\quad \frac{dT}{dx} = -mC_1 e^{-mL} + mC_2 e^{mL} = 0$

Subs. B.C.1 in B.C.2, we get

$$C_2 = \frac{\theta_0 e^{-mL}}{e^{mL} + e^{-mL}} \quad \Rightarrow C_1 = \theta_0 - \frac{\theta_0 e^{-mL}}{e^{mL} + e^{-mL}}$$

Subs. C_1&C_2 in the general equation, we obtain

$$\frac{\theta}{\theta_0} = \frac{e^{-mx}}{1+e^{-2mL}} + \frac{e^{mx}}{1+e^{2mL}}$$

$$= \frac{\cosh\,[m(L-x)]}{\cosh mL}$$

CASE 3: The fin is of finite length and loses heat by convection from its end.

B.C.1 at $x = 0$ $\quad T = T_0 \quad \theta = \theta_0$

B.C.2 at $x = L$ $\quad -k\frac{d\theta}{dx} = h\theta$

From B.C.1 $\quad \theta_0 = C_1 + C_2 \quad \Rightarrow C_1 = \theta_0 - C_2$

From B.C.2 $\quad -k\left(-mC_1 e^{-mL} + mC_2 e^{mL}\right) = h\left(C_1 e^{-mL} + C_2 e^{mL}\right)$

By applying C_1&C_2 in the general equation, we get,

$$\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_o - T_\infty} = \frac{\cosh m\,(L-x) + (h/mk)\sinh m\,(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

Calculation of Heat Lost by the Fin

The heat lost by the fin can be calculated either by

$$q = -kA\,\frac{dT}{dx}\bigg]_{x=0}$$

or by

$$q = \int_0^L hP(T - T_\infty)\,dx = \int_0^L hP\,\theta\,dx$$

FOR CASE 1:

$$\frac{\theta}{\theta_0} = e^{-mx}, \quad \frac{d\theta}{dx} = -m\theta_0 e^{-mx}, \quad \left.\frac{d\theta}{dx}\right|_{x=0} = -m\theta_0$$

$$\therefore \quad -kA(-m\theta_0) = kAm\theta_0, \quad For\ q = kA\sqrt{\frac{hP}{kA}}\theta_0 \quad \Rightarrow$$

$$q = \sqrt{hPkA}\,\theta_0$$

FOR CASE 2:

$$\frac{\theta}{\theta_0} = \frac{\cosh[m(L-x)]}{\cosh mL}, \quad \frac{d\theta}{dx} = \theta_0\left(\frac{-me^{-mx}}{1+e^{-2mx}} + \frac{me^{mx}}{1+e^{2mx}}\right)$$

$$\left.\frac{d\theta}{dx}\right|_{x=0} = \theta_0 m\left(\frac{1}{1+e^{2mx}} - \frac{1}{1+e^{-2mx}}\right) \Rightarrow \frac{d\theta}{dx} = \theta_0 m\left(-\frac{e^{mL}-e^{-mL}}{e^{mx}+e^{-mL}}\right)$$

$$\therefore \quad q = -kA\theta_0 m\left(-\frac{e^{mL}-e^{-mL}}{e^{mx}+e^{-mL}}\right) \Rightarrow$$

$$q = \sqrt{hPkA}\,\theta_0 \tanh mL$$

FOR CASE 3:

$$\frac{\theta}{\theta_0} = \frac{T - T_\infty}{T_o - T_\infty} = \frac{\cosh m\,(L-x) + (h/mk)\sinh m\,(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

The heat flow for this case is

$$q = \sqrt{hPkA}\,\theta_0\,\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$