# *Quantum Mechanics*

*Tenth Lecture*

## *Probability Current Density*

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*2022- 2023*

## Probability Current Density

- From time depend Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}+V_o\psi(x,t)=i\hbar\frac{\partial\psi(x,t)}{\partial t}\quad \ldots(1)$$

- Find the conjugate of the equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*(x,t)}{\partial x^2}+V_o\psi^*(x,t)=-i\hbar\frac{\partial\psi^*(x,t)}{\partial t}\quad \ldots(2)$$

- Now we multiply the eq. (1) by $\Psi^*$ and eq. (2) by $\Psi$ produces the following:

$$-\frac{\hbar^2}{2m}\psi^*\frac{\partial^2\psi}{\partial x^2}+V_o\psi\psi^*=i\hbar\psi^*\frac{\partial\psi}{\partial t}$$

$$-\frac{\hbar^2}{2m}\psi\frac{\partial^2\psi^*}{\partial x^2}+V_o\psi^*\psi=-i\hbar\psi\frac{\partial\psi^*}{\partial t}$$

- By subtracting the above two equations:

$$-\frac{\hbar^2}{2m}\psi^*\frac{\partial^2\psi}{\partial x^2}+V_o\psi\psi^*-\left[-\frac{\hbar^2}{2m}\psi\frac{\partial^2\psi^*}{\partial x^2}+V_o\psi^*\psi\right]=i\hbar\psi^*\frac{\partial\psi}{\partial t}-\left[-i\hbar\psi\frac{\partial\psi^*}{\partial t}\right]$$

$$-\frac{\hbar^2}{2m}\psi^*\frac{\partial^2\psi}{\partial x^2}+V_o\psi\psi^*+\frac{\hbar^2}{2m}\psi\frac{\partial^2\psi^*}{\partial x^2}-V_o\psi^*\psi=i\hbar\psi^*\frac{\partial\psi}{\partial t}+i\hbar\psi\frac{\partial\psi^*}{\partial t}$$

$$-\frac{\hbar^2}{2m}\left(\psi^*\frac{\partial^2\psi}{\partial x^2}-\psi\frac{\partial^2\psi^*}{\partial x^2}\right)=i\hbar\left(\psi^*\frac{\partial\psi}{\partial t}+\psi\frac{\partial\psi^*}{\partial t}\right)$$

$$\boxed{\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t} = \frac{\partial}{\partial t}(\psi^* \psi)}$$

$$-\frac{\hbar}{2im}\left(\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2}\right) = \frac{\partial}{\partial t}(\psi^* \psi)$$

- Notice between the large brackets in the above equation that we extract a common factor from it $\frac{\partial}{\partial x}$ and as follows:

$$-\frac{\partial}{\partial t}(\psi^* \psi) - \frac{\hbar}{2im}\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\psi^* \psi) + \frac{\hbar}{2im}\frac{\partial}{\partial x}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) = 0$$

- When comparing the last equation with the fluid flow equation:

$$\frac{\partial}{\partial t} p(x) + \frac{\partial}{\partial x} S_x = 0$$

- We will get the probability current density represented by the symbol $S_x$, which is as in the following equation:

$$S_x = \frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right) \qquad and \qquad p(x) = \psi^* \psi = |\psi|^2$$

**Example 1:** Find Probability Current Density of wave function

$$\psi(x) = Ae^{-ibx}$$

When A and b is real constants.

**Solution**

$$\psi(x) = Ae^{-ibx} \quad \Longrightarrow \quad \psi^*(x) = Ae^{+ibx}$$

$$\frac{\partial \psi}{\partial x} = -ibAe^{-ibx} \quad and \quad \frac{\partial \psi^*}{\partial x} = ibAe^{+ibx}$$

*The equation of probability current density*

$$S_x = \frac{\hbar}{2im}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

$$= \frac{\hbar}{2im}\left[Ae^{+ibx}\left(-ibAe^{-ibx}\right) - Ae^{-ibx}\left(ibAe^{+ibx}\right)\right]$$

$$S_x = \frac{\hbar}{2im}\left[-ibA^2e^{+ibx-ibx} - ibA^2e^{-ibx+ibx}\right]$$

$$S_x = \frac{\hbar}{2im}\left[-ibA^2e^0 - ibA^2e^0\right]$$

$$= -2ibA^2\frac{\hbar}{2im} = -\frac{b\hbar A^2}{m}$$

**Example 2:** prove that $m\frac{d}{dt}\langle x\rangle = \langle P_X\rangle$

$$\frac{d}{dt}\langle x\rangle = \frac{d}{dt}\int \Psi^* \, x \, \Psi \, dx = \int\left(\frac{\partial\Psi^*}{\partial t}x\Psi + \Psi^* x\frac{\partial\Psi}{\partial t}\right)dx \quad \ldots\ldots(1)$$

- From TDSE and its conjugate

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi \qquad \ldots\ldots\ldots\ldots(2)$$

$$-i\hbar\frac{\partial\Psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + V\Psi^* \qquad \ldots\ldots\ldots\ldots(3)$$

- Now we multiply the eq. (2) by $\Psi^*$x and eq. (3) by x$\Psi$ and subtract the resulting equations:

$$i\hbar\left(\Psi^* x\frac{\partial\Psi}{\partial t} + \frac{\partial\Psi^*}{\partial t}x\Psi\right) = -\left(\frac{\hbar^2}{2m}\right)\left(\Psi^* x\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}x\Psi\right) \ldots(4)$$

- Using (4) in (1)

$$\frac{d}{dt}\langle x\rangle = \frac{i\hbar}{2m}\int\left(\Psi^* x\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}x\Psi\right)dx \quad \ldots\ldots(5)$$

- The first integral can be evaluated by part:

$$\int \Psi^* x\frac{\partial^2\Psi}{\partial x^2}dx = \Psi^* x\frac{d\Psi}{dx}\Big|_0^\infty - \int\left(\frac{d\Psi}{dx}\right)\left(\Psi^* + x\frac{d\Psi^*}{dx}\right)dx \quad \ldots(6)$$

- The first term on RHS is zero at both the limits:

$$\int \Psi^* x\frac{\partial^2\Psi}{\partial x^2}dx = -\int\frac{d\Psi}{dx}\left(\Psi^* + x\frac{d\Psi^*}{dx}\right)dx$$

$$= -\int \Psi^*\frac{d\Psi}{dx}dx - \int x\left(\frac{d\Psi}{dx}\right)\left(\frac{d\Psi^*}{dx}\right)dx \qquad \ldots\ldots\ldots\ldots\ldots(7)$$

- The first integral can be evaluated by part:

$$-\int \frac{\partial^2\Psi^*}{\partial x^2}x\Psi\,dx = -\Psi x\frac{d\Psi^*}{dx}\Big|_0^\infty - \int\left(\frac{d\Psi^*}{dx}\right)\left(\Psi + x\frac{d\Psi}{dx}\right)dx$$

- The first term on RHS is zero at both the limits:

$$-\int \frac{\partial^2\Psi^*}{\partial x^2} x\Psi\, dx = -\int \frac{d\Psi^*}{dx}\left(\Psi + x\frac{d\Psi}{dx}\right) dx$$

$$= \int \Psi \frac{d\Psi^*}{dx}\, dx + \int x \left(\frac{d\Psi^*}{dx}\right)\left(\frac{d\Psi}{dx}\right) dx \qquad \ldots\ldots\ldots..(8)$$

- Substituting (7) and (8) in (5):

$$\frac{d}{dt}\langle x\rangle = \frac{i\hbar}{2m} - \int \Psi^* \frac{d\Psi}{dx} dx - \int x\left(\frac{d\Psi}{dx}\right)\left(\frac{d\Psi^*}{dx}\right) dx - \int \Psi \frac{d\Psi^*}{dx}\, dx + \int x\left(\frac{d\Psi^*}{dx}\right)\left(\frac{d\Psi}{dx}\right) dx$$

$$\frac{d}{dt}\langle x\rangle = \left(\frac{i\hbar}{2m}\right)\left(-\int \Psi^* \frac{d\Psi}{dx} dx + \int \Psi \frac{d\Psi^*}{dx}\, dx\right.$$

$$= \left(\frac{1}{2m}\right)\left(\int \Psi^*\left(-i\hbar\frac{d}{dx}\right)\Psi dx + \int \Psi\left(i\hbar\frac{d}{dx}\right)\Psi^* dx\right. \qquad \ldots\ldots\ldots.(9)$$

$\int \Psi^*\left(-i\hbar\frac{d}{dx}\right)\Psi dx = \langle P_X\rangle$ and $\int \Psi\left(i\hbar\frac{d}{dx}\right)\Psi^* dx = \langle P_X^*\rangle$

$$P_X = P_X^*$$

$$\frac{d}{dt}\langle x\rangle = \left(\frac{1}{2m}\right)\langle P_X\rangle\langle P_X\rangle = \left(\frac{1}{2m}\right)2\,\langle P_X\rangle = \left(\frac{\langle P_X\rangle}{m}\right)$$

$$m\,\frac{d}{dt}\langle x\rangle = \langle P_X\rangle$$