Quantum Mechanics

Ninth Lecture

Expectation Value and Variance

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1. Expectation Value

- If the system is in state Ψ Which is not an eigen state of a such as observable, then it is not possible to say with certainly what measure value will be found for A. Therefore, one has to use the average value which called in quantum expectation value of A.
- Expectation value is defined as

$$\langle A \rangle = \frac{\int \Psi^* \hat{A}\, \Psi\, dx}{\int \Psi^* \Psi\, dx} \qquad \ldots\ldots\ldots.(1)$$

- If Ψ is a normalized wave function, the denominator of Eq. (1) equals the probability that the particle exists somewhere between x=-∞ and x=∞ therefore has the value 1.

$$\langle A \rangle = \int \Psi^*\, \hat{A}\, \Psi\, dx \qquad \ldots\ldots\ldots(2)$$

1. Position

$$\langle x \rangle = \int \Psi^*\, \hat{x}\, \Psi\, dx \qquad \ldots\ldots\ldots.(3)$$

2. Momentum

$$\langle P \rangle = \int \Psi^*\, \hat{P}\, \Psi\, dx$$

$$\hat{p} = -i\hbar\, \frac{\partial}{\partial x}$$

$$\langle p \rangle = -i\hbar \int \Psi^*\, \frac{\partial \Psi}{\partial x}\, dx \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

3. Total energy

$$\langle E \rangle = \int \Psi^*\, \hat{E}\, \Psi\, dt$$

$$\hat{E} = i\hbar \frac{\partial}{\partial t}$$

$$\langle E \rangle = i\hbar \int \Psi^* \frac{\partial \Psi}{\partial t}\, dt \qquad \text{..........(4)}$$

4. Kinetic energy

$$\langle T \rangle = \int \Psi^* \hat{T}\, \Psi\, dx$$

$$T = \frac{\widehat{P^2}}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$\langle T \rangle = -\frac{\hbar^2}{2m}\int \Psi^* \frac{\partial^2 \Psi}{\partial x^2}\, dx \qquad \text{..........(5)}$$

Example 1: A particle limited to the x axis has the wave function $\psi = ax^2 + ibx$ between x=0 and x=1; $\psi = 0$ elsewhere. Find the expectation value <x> of the particle's position.

$$\langle \boldsymbol{x} \rangle = \int \boldsymbol{\Psi}^* \,\widehat{\boldsymbol{x}}\, \boldsymbol{\Psi}\, \boldsymbol{dx}$$

$$|\psi(\mathrm{x})|^2 = \psi^* \psi = (\mathrm{ax}^2 + \mathrm{ibx})(\mathrm{ax}^2 - \mathrm{ibx})$$
$$= \mathrm{a}^2\mathrm{x}^4 + \mathrm{b}^2\mathrm{x}^2$$
$$\therefore \quad <\mathrm{x}> = \int_0^1 \mathrm{x}\,|\psi(\mathrm{x})|^2\, \mathrm{dx}$$

$$< x > = \int_0^1 x(a^2x^4 + b^2x^2)dx$$

$$< x > = \int_0^1 (a^2x^5 + b^2x^3)dx$$

$$< x > = \left[\frac{a^2}{6}x^6 + \frac{b^2}{4}x^4\right]_0^1$$

$$< x > = \frac{a^2}{6} + \frac{b^2}{4}$$

Example 2:

A particle limited to the x axis has the wave function Ψ=ax between x=0 and x=1; Ψ= 0 elsewhere. (a) Find the probability that the particle can be found between x=0.45 and x=0.55. (b) Find the expectation value <x> of the particle's position.

(*a*) The probability is

$$\int_{x_1}^{x_2} |\Psi|^2\, dx = a^2 \int_{0.45}^{0.55} x^2 dx = a^2\left[\frac{x^3}{3}\right]_{0.45}^{0.55} = 0.0251a^2$$

(*b*) The expectation value is

$$\langle x \rangle = \int_0^1 x|\Psi|^2\, dx = a^2 \int_0^1 x^3 dx = a^2\left[\frac{x^4}{4}\right]_0^1 = \frac{a^2}{4}$$

H.W: if $\Psi = e^{-\frac{x^2}{2} - iwt}$

1. check Ψ normalized or not, then find the normalization constant

2. find the expectation of x

2. *Variance*

The deviation in the measured of the operator $\hat{A}$ from its expected value $\langle A \rangle$.

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \qquad \ldots\ldots\ldots\ldots(6)$$

Q: prove the equation of variance

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle = \int \Psi^* \ (A - \langle A \rangle)^2 \Psi \, d\tau$$

$$= \int \Psi^* \ (A^2 - 2A\langle A \rangle + \langle A \rangle^2) \ \Psi \ d\tau$$

$$= \int \Psi^* \ A^2 \ \Psi \ d\tau - \int \Psi^* \ (2A\langle A \rangle) \ \Psi \ d\tau + \int \Psi^* \ \langle A \rangle^2 \ \Psi \ d\tau$$

$$(\Delta A)^2 = \langle A^2 \rangle - 2\langle A \rangle \langle A \rangle + \langle A \rangle^2 \quad \Longrightarrow \quad (\Delta A)^2 = \langle A^2 \rangle - 2\langle A \rangle^2 + \langle A \rangle^2$$

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \qquad \text{OR} \quad (\Delta A) = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

H.W: find the variance in

1. Position $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$

2. Momentum $(\Delta P_x)^2 = \langle {P_X}^2 \rangle - \langle P_X \rangle^2$