Quantum Mechanics

Seventh Lecture

Observable and Operator

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1. What is Operators?

Any mathematical entity which act on a wave function and change it to another function.

2. Types of operator

1. Simple operator

$$\hat{A} = x \qquad \varphi_x = x^3$$

$$\hat{A}\varphi_x = x.x^3 = x^4 = \emptyset$$

2. Differential operator

$$\hat{B} = \frac{\partial}{\partial x} \qquad \widehat{E} = x\frac{\partial}{\partial x}$$

$$\hat{C} = \frac{\partial^2}{\partial x^2} \qquad \widehat{F} = \frac{\partial}{\partial x}x$$

Example1: -

Find $\widehat{F}\, f(x)$ if $\widehat{F} = \frac{\partial}{\partial x}x$

$$\widehat{F} f(x) = \frac{\partial}{\partial x} x\, f(x)$$

$$= f(x) + x\frac{\partial}{\partial x} f(x)$$

$$= \left(1 + x\frac{\partial}{\partial x}\right) f(x)$$

$$\widehat{F} = \frac{\partial}{\partial x} x = 1 + x\frac{\partial}{\partial x}$$

Example 2: -

Find $\widehat{G}\, g(x)$ if $\widehat{G} = \frac{\partial}{\partial x} x^2$

$$\widehat{G}\, g(x) = \frac{\partial}{\partial x} x^2 g(x)$$

$$= 2x\, g(x) + x^2 \frac{\partial g(x)}{\partial x}$$

$$= \left(2x + x^2 \frac{\partial g(x)}{\partial x}\right) g(x)$$

$$\frac{\partial}{\partial x} x^2 = 2x + x^2 \frac{\partial}{\partial x}$$

- ***Physical observables and their corresponding quantum operators***

Observables	Cla. Mech. Rep	Qua. Mech. Rep.
1-position	x	$\hat{X}$
2-Momentum	$p_x = m\dot{x}$	$\widehat{p_x} = -i\hbar \frac{\partial}{\partial x}$
3- kinetic energy	$T = \frac{p^2}{2m}$	$\hat{T} = \frac{\widehat{p^2}}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$
4- total energy	$E=T+V_{(x)}$	$\hat{E} = i\hbar \frac{\partial}{\partial t}$
5- Hamilton	$H = \frac{p^2}{2m} + V_{(x)}$	$\hat{H} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{(x)}$

2. Properties of operators

1-liner operator

i) $\widehat{A}(\varphi_1(x)+\varphi_2(x)) = \widehat{A}\,\varphi_1 + \widehat{A}\,\varphi_2$

ii) $\widehat{A}(a\,\varphi(x)) = a\,\widehat{A}\,\varphi(x)$

2- Commutation

$$\hat{C} = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$i)\ if\ \hat{C} = 0 \rightarrow [\hat{A}, \hat{B}] = 0$$

$$then \quad \hat{A}\hat{B} = \hat{B}\hat{A} \qquad commutation$$

$$ii)\ if\ \hat{C} = 1$$

$$\hat{C} = Unit\ operator$$

$$iii)\ if\ \hat{C} \neq 0 \rightarrow [\hat{A}, \hat{B}] \neq 0$$

$$\hat{A}\hat{B} - \hat{B}\hat{A} \neq 0$$

$$then \quad \hat{A}\hat{B} \neq \hat{B}\hat{A} \qquad not\ commutation$$

Example 3: - prove that $\left[\frac{\partial}{\partial x}, x\right]$ is unit operator

$$\hat{C} = [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$$\hat{C} = \frac{\partial}{\partial x}x - x\frac{\partial}{\partial x} \qquad multiplyed\ \varphi_x$$

$$\hat{C}\varphi_x = [\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}]\varphi_x$$

$$\hat{C}\varphi_x = \left[\frac{\partial}{\partial x}x(\varphi_x) - x\frac{\partial}{\partial x}(\varphi_x)\right]$$

$$= \varphi_{(x)}\frac{\partial x}{\partial x} + x\frac{\partial \varphi_{(x)}}{\partial x} - x\frac{\partial \varphi_{(x)}}{\partial x}$$

$$\hat{c}\varphi_x = \varphi_x$$

Then $C = \frac{\varphi_x}{\varphi_x}$ then C =1

Example 4: - prove that $[\hat{x},\hat{p}] = i\hbar$

$[\hat{x},\hat{p}]\ \Psi = [x\ p_x - p_x\ x]\ \Psi = [-\ x i\hbar \frac{\partial}{\partial x} - (-i\hbar \frac{\partial}{\partial x} x)]\ \Psi$

$$[xp_x - p_x x]\ \Psi = -i\hbar \left[x\frac{\partial \Psi}{\partial x} - \frac{\partial}{\partial x}(x\Psi) \right]$$

$$[xp_x - p_x x]\ \Psi = i\hbar\Psi$$

$[\hat{x},\hat{p}]\ \Psi = i\hbar\ \Psi \qquad \therefore\ [\hat{x},\hat{p}] = i\hbar$

Example 5: - Find

$$[x^n, p_x] f(x)$$

$$[x^n, p_x] f(x) = -i\hbar x^n \frac{df(x)}{dx} + i\hbar \frac{d}{dx} x^n f(x)$$

$$[x^n, p_x] f(x) = -i\hbar x^n \frac{df(x)}{dx} + i\hbar n\ x^{n-1} f(x) + i\hbar x^n \frac{df(x)}{dx}$$

$$[x^n, p_x] f(x) = i\hbar n\ x^{n-1} f(x)$$

H.W: - Find 1.

$$[y, p_y] = [z, p_z] = i\hbar$$

2.

$$[x, p^2] = 2i\hbar p$$

3.

$$[p_x, x^n] f(x)$$

3. Eigen functions and Eigen values

When an operator on a function the outcome is another functions.

$$\hat{A}f = kf$$

Example 6: - prove that the function

$$\psi = Ae^{-\alpha x}$$

is an eigen function of the operator

$$\hat{F} = \frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} + \frac{2\alpha}{x}$$

Solve:

$$\hat{F}\psi = \frac{d^2}{dx^2}\left(Ae^{-\alpha x}\right) + \frac{2}{x}\frac{d}{dx}\left(Ae^{-\alpha x}\right) + \frac{2\alpha}{x}\left(Ae^{-\alpha x}\right)$$

$$\hat{F}\psi = \alpha^2 Ae^{-\alpha x} + \frac{2}{x}\left(-\alpha Ae^{-\alpha x}\right) + \frac{2\alpha}{x}\left(Ae^{-\alpha x}\right)$$

$$\hat{F}\psi = \left(\alpha^2 - \frac{2\alpha}{x} + \frac{2\alpha}{x}\right)Ae^{-\alpha x}$$

$$\hat{F}\psi = \alpha^2 Ae^{-\alpha x}$$

$$\hat{F}\psi = \alpha^2\psi$$

H.W: - by using the eigen value equation show that the function $\varphi_n(x) = e^{i4x}$ is an eigen function of the operator $\hat{A} = \frac{\partial}{\partial x}$

H.W: - by using the eigen value equation show that the function $\varphi_n(x)$ = sin(6x) is an eigen function of the operator $\hat{A} = \frac{\partial^2}{\partial x^2}$