Analytic Mechanics

Seventh lecture

The simple harmonic motion

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Second Stage

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2022- 2023

1.1 Linear Restoring Force. Harmonic Motion

One of the most important cases of rectilinear motion, from a practical as well as from a theoretical standpoint, is that produced by a *linear restoring force*. This is a force whose magnitude is proportional to the displacement of a particle from some equilibrium position and whose direction is always opposite to that of the displacement. Such a force is exerted by an elastic cord or by a spring obeying Hooke's law

$$F = -k(X - a) = -kx \tag{1}$$

where X is the total length, and a is the unstretched (zero load) length of the spring. The variable $x = X - a$ is the displacement of the spring from its equilibrium length. The proportionality constant k is called the *stiffness*. Let a particle of mass m be attached to the spring, as shown in Figure 2.4(a); the force acting on the particle is that given by Equation 1 Let the

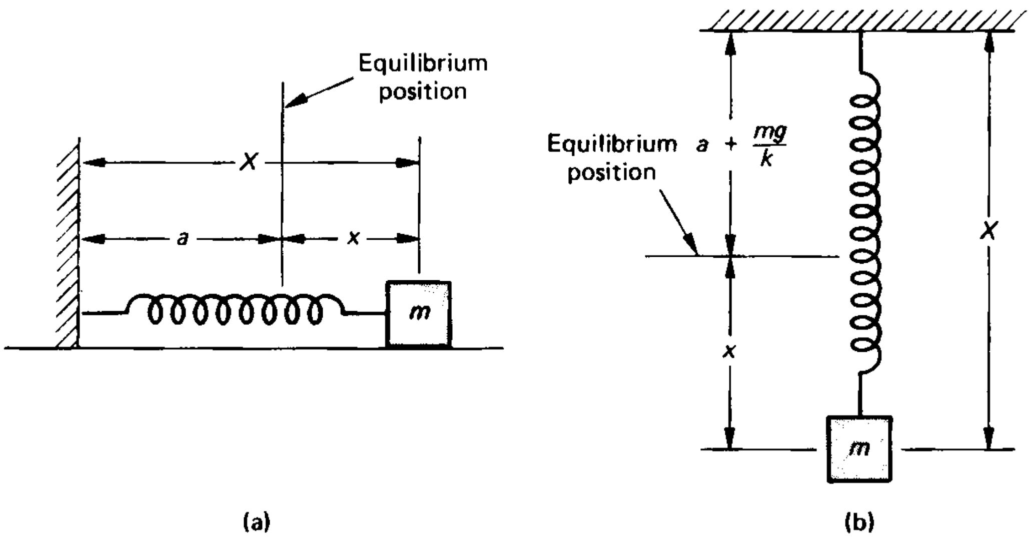

FIGURE 2.4 Illustrating the linear harmonic oscillator by means of a block of mass m and a spring. (a) Horizontal motion; (b) vertical motion.

same spring be held vertically, supporting the same particle, as shown in Figure 2.4(b). The total force now acting on the particle is

$$F = -k(X - a) + mg \tag{1}$$

where the positive direction is downward. Now, in the latter case, let us measure x relative to the new equilibrium position; that is, let $x = X - a - mg/k$. This gives again $F = -kx$, and so the differential equation of motion in either case is

$$m\ddot{x} + kx = 0 \tag{2}$$

The above differential equation of motion is met in a wide variety of physical problems. In the particular example that we are using here, the constants m and k refer to the mass of a body and to the stiffness of a spring, respectively, and the displacement x is a distance. The same equation is encountered, as we shall see later, in the case of a pendulum, where the displacement is an angle, and where the constants involve the acceleration of gravity and the length of the pendulum. Again, in certain types of electrical circuits, this equation is found to apply, where the constants represent the circuit parameters, and the quantity x represents electric current or voltage.

Equation (2) can be solved in a number of ways. It is one example of an important class of differential equations known as *linear differential equations with constant coefficients*.[8] Many, if not most, of the differential equations of physics are second-order linear differential equations. To solve Equation (2) we shall employ the trial method in which the function Ae^{qt} is the trial solution where q is a constant to be determined. If $x = Ae^{qt}$ is, in fact, a solution, then for all values of t we must have

$$m\frac{d^2}{dt^2}(Ae^{qt}) + k(Ae^{qt}) = 0$$

which reduces, upon canceling the common factors, to the equation[9]

$$mq^2 + k = 0$$

that is

$$q = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

[8] The general nth-order equation of this type is

$$c_n\frac{d^nx}{dt^n} + \cdots + c_2\frac{d^2x}{dt^2} + c_1\frac{dx}{dt} + c_0 = b(t)$$

The equation is called *homogeneous* if $b = 0$.

[9] This equation is called the *auxiliary equation*.

where $i = \sqrt{-1}$, and $\omega_0 = \sqrt{k/m}$. Now, for linear differential equations, solutions are additive. (That is, if f_1 and f_2 are solutions then the sum $f_1 + f_2$ is also a solution.) The general solution of Equation 2) is then

$$x = A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t} \qquad 3\ )$$

Since $e^{iu} = \cos u + i \sin u$, alternate forms of the solution are

$$x = a \sin \omega_0 t + b \cos \omega_0 t \qquad 4$$

or

$$x = A \cos (\omega_0 t + \theta_0) \qquad 5$$

The constants of integration in the above solutions are determined from the initial conditions. That all three expressions are solutions of Equation (2.34) may be verified by direct substitution. The motion is a sinusoidal oscillation of the displacement x. For this reason Equation (' is often referred to as the differential equation of the *harmonic oscillator* or the *linear oscillator*.

The coefficient ω_0 is called the *angular frequency*. The maximum value of x is called the *amplitude* of the oscillation; it is the constant A in Equation (, or $(a^2 + b^2)^{1/2}$ in Equation (4). The period T_0 of the oscillation is the time required for one complete cycle, as shown in Figure 2.5; that is,

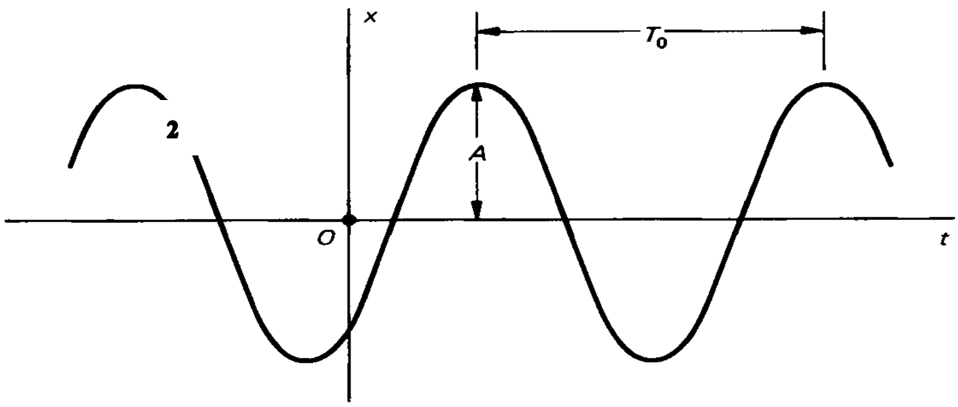

FIGURE 2.5 Graph of displacement versus time for the harmonic oscillator.

the period is the time for which the product ωt increases by just 2π, thus

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m}{k}} \qquad 6$$

The *linear frequency* of oscillation f_0 is defined as the number of cycles in unit time, therefore

$$\omega_0 = 2\pi f_0$$

$$f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{7}$$

It is common usage to employ the word "frequency" for either the angular or the linear frequency; which one is meant is usually clear from context.

EXAMPLE

A light spring is found to stretch an amount b when it supports a block of mass m. If the block is pulled downward a distance l from its equilibrium position and released at time $t = 0$, find the resulting motion as a function of t. First, to find the spring stiffness, we note that in the static equilibrium condition

$$F = -kb = -mg$$

so that

$$k = \frac{mg}{b}$$

Hence the angular frequency of oscillation is

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{b}}$$

In order to find the constants for the equation of motion

$$x = A\cos(\omega_0 t + \theta_0)$$

we have

$$x = l \qquad \text{and} \qquad \dot{x} = 0$$

at time $t = 0$. But

$$\dot{x} = -A\omega_0 \sin(\omega_0 t + \theta_0)$$

Thus

$$A = l \qquad \theta_0 = 0$$

so

$$x = l\cos\left(\sqrt{\frac{g}{b}}\,t\right)$$

is the required expression.

1.2 Energy Considerations in Harmonic Motion

Consider a particle moving under a linear restoring force $F = kx$. Let us calculate the work W done by an external force F_a in moving the particle from the equilibrium position ($x = 0$) to some position x. We have $F_a = -F = kx$, and so

$$W = \int F_a \, dx = \int_0^x (kx) \, dx = \frac{k}{2} x^2$$

The work W is stored in the spring as potential energy

$$V(x) = W = \frac{k}{2} x^2 \qquad 8$$

Thus $F = -dV/dx = -kx$ as required by the definition of V, Equation (2.13). The total energy E is then given by the sum of the kinetic and potential energies as

$$E = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 \qquad 9$$

We can now solve for the velocity as a function of displacement

$$\dot{x} = \left(\frac{2E}{m} - \frac{k}{m} x^2\right)^{1/2}$$

This can be integrated to give t as a function of x as follows:

$$t = \int \frac{dx}{\sqrt{(2E/m) - (k/m)x^2}} = \sqrt{\frac{m}{k}} \cos^{-1}\left(\frac{x}{A}\right) + C$$

in which

$$A = \sqrt{\frac{2E}{k}}$$

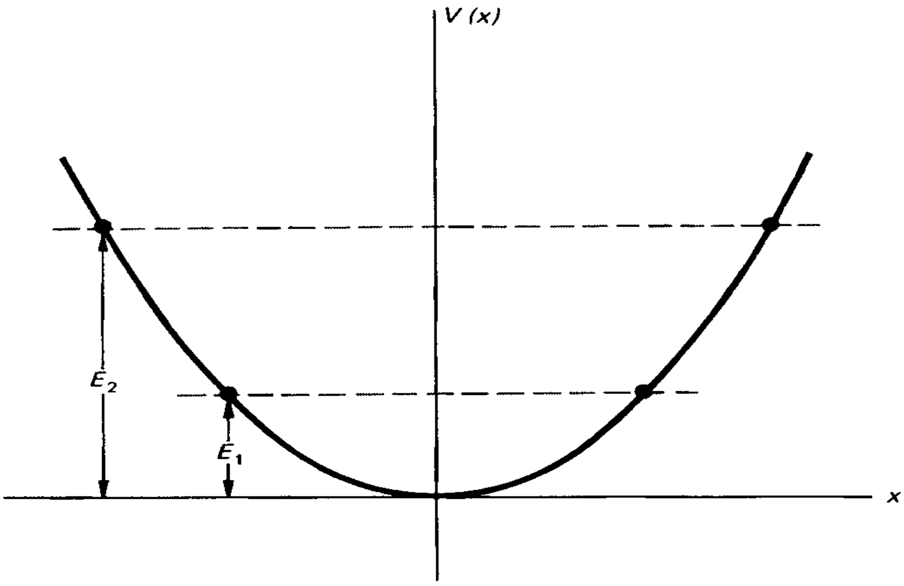

FIGURE 2.6 Graph of the potential energy function of the harmonic oscillator. The turning points defining the amplitude are shown for two values of the total energy.

and C is a constant of integration. Upon solving the integrated equation for x as a function of t, we find the very same relationship as that found in the previous section, except that we now obtain an explicit value for the amplitude A. We could also have found the amplitude directly from the energy equation 9 by noting that x must lie between $\sqrt{2E/k}$ and $-\sqrt{2E/k}$ in order for $\dot{x}$ to be real. This is illustrated in Figure 2.6 which shows the potential energy function and the turning points of the motion for different values of the total energy E.

From the energy equation we see that the maximum value of $\dot{x}$, which we shall call v_{max}, occurs when $x = 0$, and so we have

$$E = \tfrac{1}{2}mv_{max}^2 = \tfrac{1}{2}kA^2$$

or

$$v_{max} = \sqrt{\frac{k}{m}}\,A = \omega_0 A$$