# *Analytic Mechanics*

***Eighth lecture***

## *Action potential and electrical activity of neurons*

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# 1. Motion of Charged Particles in Electric and Magnetic Fields

For a suitable choice of axes, the differential equations for the nonisotropic case can be written

$$
\begin{aligned}
m\ddot{x} &= -k_1 x \\
m\ddot{y} &= -k_2 y \\
m\ddot{z} &= -k_3 z
\end{aligned}
$$

Here we have a case of *three* different frequencies of oscillation: $\omega_1 = \sqrt{k_1/m}$, $\omega_2 = \sqrt{k_2/m}$, $\omega_3 = \sqrt{k_3/m}$, and the motion is given by the solutions

$$
\begin{aligned}
x &= A \cos(\omega_1 t + \alpha) \\
y &= B \cos(\omega_2 t + \beta) \\
z &= C \cos(\omega_3 t + \gamma)
\end{aligned}
$$

Again, the six constants of integration in the above equations are determined from the initial conditions. The resulting oscillation of the particle lies entirely within a rectangular box (whose sides are $2A$, $2B$, and $2C$) centered on the origin. In the event that $\omega_1$, $\omega_2$, and $\omega_3$ are commensurate, that is, if

$$\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2} = \frac{\omega_3}{n_3} \qquad 1$$

where $n_1$, $n_2$, and $n_3$ are integers, the path, called a *Lissajous* figure, will be closed, because after a time $2\pi n_1/\omega_1 = 2\pi n_2/\omega_2 = 2\pi n_3/\omega_3$ the particle will return to its initial position and the motion will be repeated. [In Equation it is assumed that any common integral factor is canceled out.] On the other hand, if the $\omega$'s are *not* commensurate, the path is not closed. In this case the path may be said to fill completely the rectangular box mentioned above, at least in the sense that if we wait long enough, the particle will come arbitrarily close to any given point.

The net restoring force exerted on a given atom in a solid crystalline substance is approximately linear in the displacement in many cases. The resulting frequencies of oscillation usually lie in the infrared region of the spectrum: $10^{12}$ to $10^{14}$ vibrations per second.

## 2.Motion of Charged Particles in Electric and Magnetic Fields

When an electrically charged particle is in the vicinity of other electric charges, it experiences a force. This force $\mathbf{F}$ is said to be due to the electric field $\mathbf{E}$ which arises from these other charges. We write

$$\mathbf{F} = q\mathbf{E}$$

where $q$ is the electric charge carried by the particle in question.[2] The equation of motion of the particle is then

$$m\frac{d^2\mathbf{r}}{dt^2} = q\mathbf{E}$$

or, in component form,

$$m\ddot{x} = qE_x$$
$$m\ddot{y} = qE_y$$
$$m\ddot{z} = qE_z$$

The field components are, in general, functions of the position coordinates $x$, $y$, and $z$. In the case of time-varying fields (that is, if the charges producing $\mathbf{E}$ are moving) the components, of course, also involve $t$.

Let us consider a simple case, namely that of a uniform constant electric field. We can choose one of the axes, say the $z$ axis, to be in the direction of the field. Then $E_x = E_y = 0$, and $E = E_z$. The differential equations of motion of a particle of charge $q$ moving in this field are then

$$\ddot{x} = 0 \qquad \ddot{y} = 0 \qquad \ddot{z} = \frac{qE}{m} = \text{constant}$$

These are of exactly the same form as those for a projectile in a uniform gravitational field. The path is therefore a parabola.

It is shown in textbooks dealing with electromagnetic theory[3] that

$$\nabla \times \mathbf{E} = 0$$

if $\mathbf{E}$ is due to static charges. This means that motion in such a field is conservative, and that there exists a potential function $\Phi$ such that $\mathbf{E} = -\nabla\Phi$. The potential energy of a particle of charge $q$ in such a field is then $q\Phi$, and the total energy is constant and is equal to $\frac{1}{2}mv^2 + q\Phi$.

In the presence of a static magnetic field $\mathbf{B}$ (called the magnetic induction) the force acting on a moving particle is conveniently expressed by means of the cross product, namely,

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

where $\mathbf{v}$ is the velocity, and $q$ is the charge.[4] The differential equation of motion of a particle moving in a purely magnetic field is then

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{v} \times \mathbf{B})$$

The above equation states that the acceleration of the particle is always at right angles to the direction of motion. This means that the tangential component of the acceleration ($\dot{v}$) is zero, and so the particle moves with constant speed. This is true even if $\mathbf{B}$ is a varying function of the position $\mathbf{r}$ as long as it does not vary with time.

## EXAMPLE

Let us examine the motion of a charged particle in a uniform constant magnetic field. Suppose we choose the $z$ axis to be in the direction of the field; that is, we shall write

$$\mathbf{B} = \mathbf{k}B$$

The differential equation of motion now reads

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{v} \times \mathbf{k}B) = qB\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & 1 \end{vmatrix}$$

$$m(\mathbf{i}\ddot{x} + \mathbf{j}\ddot{y} + \mathbf{k}\ddot{z}) = qB(\mathbf{i}\dot{y} - \mathbf{j}\dot{x})$$

Equating components, we have

$$m\ddot{x} = qB\dot{y}$$
$$m\ddot{y} = -qB\dot{x}$$
$$\ddot{z} = 0$$

Here, for the first time, we meet a set of differential equations of motion which are *not* of the separated type. The solution is relatively simple, however, for we can integrate at once with respect to $t$ to obtain

$$m\dot{x} = qBy + c_1$$
$$m\dot{y} = -qBx + c_2$$
$$\dot{z} = \text{constant} = \dot{z}_0$$

or

$$\dot{x} = \omega y + C_1 \qquad \dot{y} = -\omega x + C_2 \qquad \dot{z} = \dot{z}_0$$ @

where we have used the abbreviation $\omega = qB/m$. The $c$'s are constants of integration, and $C_1 = c_1/m$, $C_2 = c_2/m$. Upon inserting the expression for $\dot{y}$ from the second part of Equation @ into the first part of Equation ), we obtain the following separated equation for $x$:

$$\ddot{x} + \omega^2 x = \omega^2 a$$

where $a = C_2/\omega$. The solution is clearly

$$x = a + A \cos(\omega t + \theta_0) \qquad 1$$

where $A$ and $\theta_0$ are constants of integration. Now, if we differentiate with respect to $t$, we have

$$\dot{x} = -A\omega \sin(\omega t + \theta_0) \qquad *$$

The above expression for $\dot{x}$ may be substituted for the left side of the first of Equations ( ) and the resulting equation solved for $y$. The result is

$$y = b - A \sin(\omega t + \theta_0) \qquad 2$$

where $b = -C_1/\omega$. To find the form of the path of motion, we eliminate $t$ between Equation 1 and Equation 2 to get

$$(x - a)^2 + (y - b)^2 = A^2 \qquad 3$$

Thus the projection of the path of motion on the $xy$ plane is a circle of radius

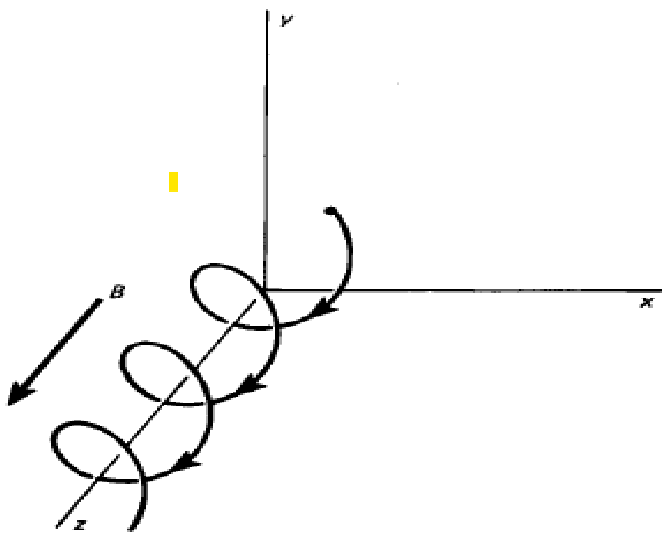


FIGURE 1 Helical path of a charged particle moving in a magnetic field.

$A$ centered at the point $(a,b)$. Since, from the third of Equations @ the speed in the $z$ direction is constant, we conclude that the path is a *spiral*. The axis of the spiral path is in the direction of the magnetic field, as shown in Figure 1 From Equation * we have

$$\dot{y} = -A\omega \cos(\omega t + \theta_0) \qquad **$$

Upon eliminating $t$ between Equation ( * and Equation (3.40), we find

$$\dot{x}^2 + \dot{y}^2 = A^2\omega^2 = A^2\left(\frac{qB}{m}\right)^2 \qquad **$$

Letting $v_1 = (\dot{x}^2 + \dot{y}^2)^{1/2}$, we see that the radius $A$ of the spiral is given by

$$A = \frac{v_1}{\omega} = v_1 \frac{m}{qB}$$

If there is no component of the velocity in the $z$ direction, the path is a of radius $A$. It is evident that $A$ is directly proportional to the speed $v_1$, and that the angular frequency $\omega$ of motion in the circular path is *independent* of the speed. $\omega$ is known as the cylotron frequency. The cyclotron, invented by Ernest Lawrence, depends for its operation on the fact that $\omega$ is independent of the speed of the charged particle.

# 3.Constrained Motion of a Particle

When a moving particle is restricted geometrically in the sense that it must stay on a certain definite surface or curve, the motion is said to be *constrained*. A piece of ice sliding around in a bowl, or a bead sliding on a wire, are examples of constrained motion. The constraint may be complete, as with the bead, or it may be one sided, as in the former example. Constraints may be fixed, or they may be moving. In this chapter we shall study only fixed constraints.

*The Energy Equation for Smooth Constraints*

The total force acting on a particle moving under constraint can be expressed as the vector sum of the external force **F** and the force of constraint **R**. The latter force is the reaction of the constraining agent upon the particle. The equation of motion may therefore be written

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{R}$$

If we take the dot product with the velocity $\mathbf{v}$ we have

$$m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v} + \mathbf{R}\cdot\mathbf{v}$$

Now in the case of a *smooth* constraint—for example, a frictionless surface—the reaction **R** is normal to the surface or curve while the velocity **v** is tangent to the surface. Hence **R** is perpendicular to **v** and the dot product $\mathbf{R}\cdot\mathbf{v}$ vanishes. Equation (3.44) then reduces to

$$\frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) = \mathbf{F}\cdot\mathbf{v}$$

Consequently, if **F** is conservative, we can integrate as in Section 3.5, and we find the same energy relation as Equation

$$\tfrac{1}{2}mv^2 + V(x,y,z) = \text{constant} = E$$

Thus the particle, although remaining on the surface or curve, moves in such a way that the total energy is constant. We might, of course, have expected this to be the case for frictionless constraints.

## EXAMPLE

A particle is placed on top of a smooth sphere of radius $a$. If the particle is slightly disturbed, at what point will it leave the sphere?

The forces acting on the particle are the downward force of gravity and the reaction **R** of the spherical surface. The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + \mathbf{R}$$

Let us choose coordinate axes as shown in Figure 3.8. The potential energy is then $mgz$, and the energy equation reads

$$\tfrac{1}{2}mv^2 + mgz = E$$

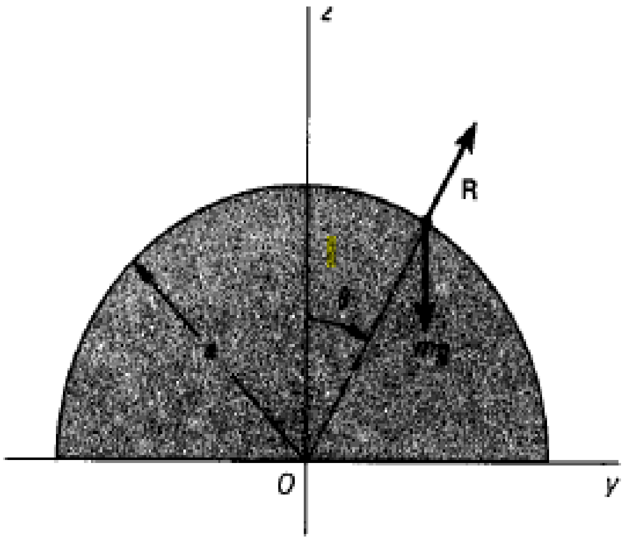


FIGURE 2 Forces acting on a particle sliding on a smooth sphere.

From the initial conditions ($v = 0$ for $z = a$) we have $E = mga$, so, as the particle slides down, its speed is given by the equation

$$v^2 = 2g(a - z)$$

Now, if we take radial components of the equation of motion, we can write the force equation as

$$-\frac{mv^2}{a} = -mg\cos\theta + R = -mg\frac{z}{a} + R$$

Hence

$$R = mg\frac{z}{a} - \frac{mv^2}{a} = mg\frac{z}{a} - \frac{m}{a}2g(a - z)$$

$$= \frac{mg}{a}(3z - 2a)$$

Thus $R$ vanishes when $z = \frac{2}{3}a$, at which point the particle will leave the sphere. This may be argued from the fact that the sign of $R$ changes from positive to negative there.

*Motion on a Curve*

For the case in which a particle is constrained to move on a certain curve, the energy equation together with the equations of the curve in parametric form

$$x = x(s) \qquad y = y(s) \qquad z = z(s)$$

suffice to determine the motion. (The parameter $s$ is the distance measured along the curve from some arbitrary reference point.) The motion may be found by consideration of the fact that the potential energy can be expressed as a function of $s$ alone, while the kinetic energy is just $\frac{1}{2}m\dot{s}^2$. Thus the energy equation may be written

$$\tfrac{1}{2}m\dot{s}^2 + V(s) = E$$

from which $s$ (hence $x$, $y$, and $z$) can be obtained by integration. Alternately, by differentiating the above equation with respect to $t$ and canceling the common factor $\dot{s}$, we obtain the following differential equation of motion for the particle:

$$m\ddot{s} + \frac{dV}{ds} = 0$$

This equation is equivalent to the equation

$$m\ddot{s} - F_s = 0$$

where $F_s$ is the component of the external force $\mathbf{F}$ in the direction of $s$. This means that $F_s = -dV/ds$.