Analytic Mechanics

Ninth lecture

Simple pendulum

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1. The Simple Pendulum

The above considerations are well illustrated by the simple pendulum—a heavy particle attached to the end of a light inextensible rod or cord, the motion being in a vertical plane. The simple pendulum is also dynamically equivalent to a bead sliding ... smooth wire in the form of a vertical circular loop. As shown in Figure 1 let θ be the angle between the vertical and

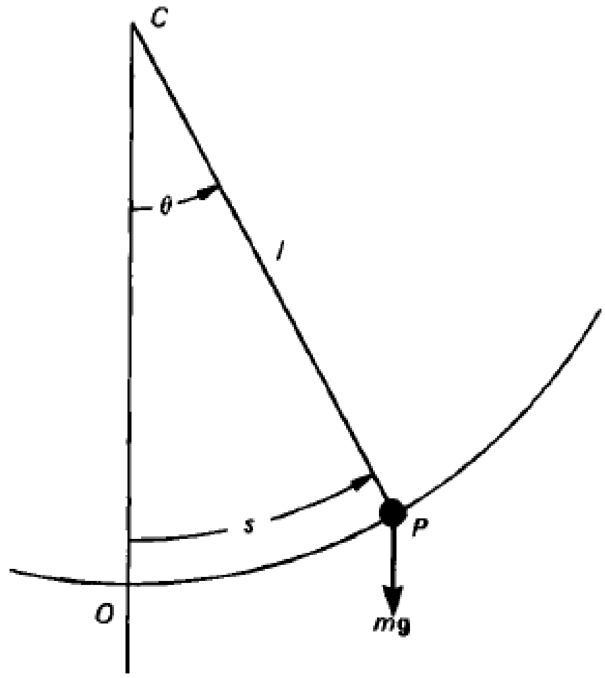

FIGURE 1 The simple pendulum.

In order to find an approximate solution of the differential equation of motion, let us assume that θ remains small. In this case

$$\sin \theta \simeq \theta$$

so we have

$$\ddot{\theta} + \frac{g}{l}\,\theta = 0$$

This is the differential equation of the harmonic oscillator. The solution, as

$$\theta = \theta_0 \cos(\omega_0 t + \varphi_0)$$

where $\omega_0 = \sqrt{g/l}$. θ_0 is the amplitude of oscillation, and φ_0 is a phase factor. Thus, to the extent that θ is a valid approximation for $\sin\theta$, the motion is simple harmonic, and the period of oscillation T_0 is given by

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}$$

the well-known elementary formula.

the line CP where C is the center of the circular path and P is the instantaneous position of the particle. The distance s is measured from the equilibrium position O. From the figure, we see that the component F_s of the force of gravity $m\mathbf{g}$ in the direction of s is equal to $-mg\sin\theta$. If l is the length of the pendulum, then $\theta = s/l$. The differential equation of motion then reads

$$m\ddot{s} + mg\sin\left(\frac{s}{l}\right) = 0$$

or, in terms of θ, we may write

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

It should be noted that the potential energy V can be expressed as mgz where z is the vertical distance of the particle from O, namely,

$$V = mgz = mgl(1 - \cos\theta)$$
$$= mgl - mgl\cos\left(\frac{s}{l}\right)$$

Hence $-dV/ds = -mg\sin(s/l) = -mg\sin\theta = F_s$.

2. More Accurate Solution of the Simple Pendulum Problem and the Nonlinear Oscillator

The differential equation of motion of the simple pendulum

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

is a special case of the general differential equation for motion under a nonlinear restoring force, that is, a force which varies in some manner other than in direct proportion to the displacement. The equation of the general one-dimensional problem with no damping may be written

$$\ddot{\xi} + f(\xi) = 0$$

where ξ is the variable denoting the displacement from the equilibrium position, so that

$$f(0) = 0$$

Nonlinear differential equations usually require some method of approximation for their solution. Suppose that the function $f(\xi)$ is expanded as a power series in ξ, namely,

$$f(\xi) = a_1\xi + a_2\xi^2 + a_3\xi^3 + \cdots$$

The differential equation of motion is then

$$\frac{d^2\xi}{dt^2} + a_1\xi + a_2\xi^2 + a_3\xi^3 + \cdots = 0$$

This is the expanded form of the general equation of motion of the nonlinear oscillator without damping. The term $a_1\xi$ in the above equation is the *linear* term. If this term is predominant, that is, if a_1 is much larger than the other coefficients, then the motion will be approximately simple harmonic with angular frequency $a_1^{1/2}$. A more accurate solution must take into account the remaining nonlinear terms.

To illustrate, let us return to the problem of the simple pendulum. If we use the series expansion

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

and retain only the first two terms, we obtain

$$\ddot{\theta} + \frac{g}{l}\,\theta - \frac{g}{6l}\,\theta^3 = 0$$

as a second approximation to the differential equation of motion. We know that the motion is periodic. Suppose we *try* a solution in the form of a simple sinusoidal function

$$\theta = A \cos \omega t$$

Inserting this into the differential equation, we obtain

$$-A\omega^2 \cos \omega t + \frac{g}{l}\,A \cos \omega t - \frac{g}{6l}\,A^3 \cos^3 \omega t = 0$$

or, upon using the trigonometric identity

$$\cos^3 u = \tfrac{3}{4} \cos u + \tfrac{1}{4} \cos 3u$$

we have, after collecting terms,

$$\left(-A\omega^2 + \frac{g}{l}\,A - \frac{gA^3}{8l}\right) \cos \omega t - \frac{gA^3}{24l} \cos 3\omega t = 0$$

Excluding the trivial case $A = 0$, we see that the above equation cannot hold for all values of t. Hence our trial function $A \cos \omega t$ cannot be a solution. From the fact that the term in $\cos 3\omega t$ appears in the above equation, however, we might suspect that a trial solution of the form 1

$$\theta = A \cos \omega t + B \cos 3\omega t$$

will represent a better approximation than $A \cos \omega t$. This turns out to be the case. If we insert the above solution into Equation (3.49), we find, after a procedure similar to that above, the following equation:

$$\left(-A\omega^2 + \frac{g}{l}\,A - \frac{gA^3}{8l}\right) \cos \omega t + \left(-9B\omega^2 + \frac{g}{l}\,B - \frac{gA^3}{24l}\right) \cos 3\omega t$$
$$+ \text{ (terms in higher powers of } B \text{ and higher multiples of } \omega t) = 0$$

Again the equation will not hold for all values of t, but our approximate solution will be reasonably accurate if the coefficients of the first two cosine terms can be made to vanish separately:

$$-A\omega^2 + \frac{g}{l}A - \frac{gA^3}{8l} = 0 \qquad -9B\omega^2 + \frac{g}{l}B - \frac{gA^3}{24l} = 0$$

From the first equation

$$\omega^2 = \frac{g}{l}\left(1 - \frac{A^2}{8}\right) \qquad 2$$

With this value of ω^2, we find from the second equation

$$B = -A^3 \frac{1}{3(64 + 27A^2)} \simeq -\frac{A^3}{192}$$

Now, from our trial solution Equation 1 we see that the amplitude θ_0 of the oscillation of the pendulum is given by

$$\theta_0 = A + B$$
$$= A - \frac{A^3}{192}$$

or, if A is small,

$$\theta_0 \simeq A$$

The meaning of Equation 2 is now clear. The frequency of oscillation depends on the amplitude θ_0. In fact, we can write

$$\omega \simeq \sqrt{\frac{g}{l}}\left(1 - \frac{1}{8}\theta_0^2\right)^{1/2}$$

or, for the period, we have

$$T = \frac{2\pi}{\omega} \simeq 2\pi\sqrt{\frac{l}{g}}\left(1 - \frac{1}{8}\theta_0^2\right)^{-1/2}$$
$$\simeq 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{1}{16}\theta_0^2 + \cdots\right)$$
$$\simeq T_0\left(1 + \frac{1}{16}\theta_0^2 + \cdots\right)$$

where T_0 is the period for zero amplitude.

The above analysis, although it is admittedly very crude, brings out two essential features of free oscillation under a nonlinear restoring force; that is, the period of oscillation is a function of the amplitude of vibration, and the oscillation is not strictly sinusoidal but can be considered as the superposition of a mixture of harmonics. It can be shown that the vibration of a nonlinear system driven by a purely sinusoidal driving force will also be distorted; that is, it will contain harmonics. The loudspeaker of a radio receiver or a "hi-fi" system, for example, may introduce distortion (harmonics) over and above that introduced by the electronic amplifying system.

3 Exact Solution of the Motion of the Simple Pendulum by Means of Elliptic Integrals

From the expression for the potential energy of the simple pendulum we can write the energy equation as follows:

$$\tfrac{1}{2}m(l\dot{\theta})^2 + mgl(1 - \cos\theta) = E$$

If the pendulum is pulled aside at an angle θ_0 (the amplitude) and released ($\dot{\theta}_0 = 0$), then $E = mgl(1 - \cos\theta_0)$. The above equation then reduces to

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0)$$

after transposing terms and dividing by ml^2. By use of the identity $\cos\theta = 1 - 2\sin^2(\theta/2)$, we can further write

$$\dot{\theta}^2 = \frac{4g}{l}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right)$$

It is expedient to express the motion in terms of the variable φ defined by the equation

$$\sin\varphi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} = \frac{1}{k}\sin\frac{\theta}{2} \qquad 3$$

Upon differentiating with respect to t, we have

$$(\cos\varphi)\dot{\varphi} = \frac{1}{k}\cos\left(\frac{\theta}{2}\right)\frac{\dot{\theta}}{2} \qquad 4$$

From Equations 3 and 4 we can readily transform Equation (3.55) into the corresponding equation in φ, namely,

$$\dot{\varphi}^2 = \frac{g}{l}(1 - k^2\sin^2\varphi)$$

The relationship between φ and t is then found by separating variables and integrating:

$$t = \sqrt{\frac{l}{g}}\int_0^{\varphi}\frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}} = \sqrt{\frac{l}{g}}F(k,\varphi)$$

The function $F(k,\varphi) = \int_0^{\varphi} (1 - k^2 \sin^2 \varphi)^{-1/2}\, d\varphi$ is known as the *incomplete elliptic integral of the first kind.* The period of the pendulum is obtained by noting that θ increases from 0 to θ_0 in one quarter of a cycle. Thus we see that φ goes from 0 to $\pi/2$ in the same time interval. Therefore, we may write for the period T

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = 4\sqrt{\frac{l}{g}}\, K(k) \qquad 5$$

The function $K(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 \varphi)^{-1/2}\, d\varphi = F(k,\pi/2)$ is called the *complete elliptic integral of the first kind.* Values of the elliptic integrals are tabulated.[5] An approximate expression may be obtained, however, by expanding the integrand in Equation by the binomial theorem and integrating term by term. The result 5

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \left(1 + \frac{k^2}{2} \sin^2 \varphi + \cdots\right) d\varphi = 2\pi\sqrt{\frac{l}{g}} \left(1 + \frac{k^2}{4} + \cdots\right)$$

Now, for small values of the amplitude θ_0, we have

$$k^2 = \sin^2 \frac{\theta_0}{2} \simeq \frac{\theta_0^2}{4}$$

Thus we may write approximately

$$T \simeq 2\pi\sqrt{\frac{l}{g}} \left(1 + \frac{\theta_0^2}{16} + \cdots\right)$$

which agrees with the value of T found in the previous section.

EXAMPLE

Find the period of a simple pendulum swinging with an amplitude of 20°. Use tables of elliptic functions, and also compare with the values calculated by the above approximations.

For an amplitude of 20°, $k = \sin 10° = 0.17365$, and $\theta_0/2 = 0.17453$ radians. The results are as follows:

From tables and Equation (3.60) $T = 4\sqrt{l/g}\, K(10°) = \sqrt{l/g}\, (6.3312)$
From Equation (3.61) $T = 2\pi\sqrt{l/g}\, (1 + \frac{1}{4}\sin^2 10°) = \sqrt{l/g}\, (6.3306)$
From Equation (3.62) $T = 2\pi\sqrt{l/g}\, (1 + \theta_0^2/16) = \sqrt{l/g}\, (6.3310)$
Elementary formula $T_0 = 2\pi\sqrt{l/g} = \sqrt{l/g}\, (6.2832)$