Analytic Mechanics

Third lecture

Structure of the atom

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1. *Introduction*

Coordinate systems is an artificial mathematical tool that used to describe the position of an object in space.. There are three coordinate systems:

1. One dimension coordinate system (1D).
2. Two dimension coordinate system (2D).
3. Three dimension coordinate system (3D).

2. (1D) Coordinate system

The easiest coordinate system use to describe the location of objects in one dimensional space. For example, to describe the location of a train along a straight section of track that runs in the East-West direction. {Figure (1)}.

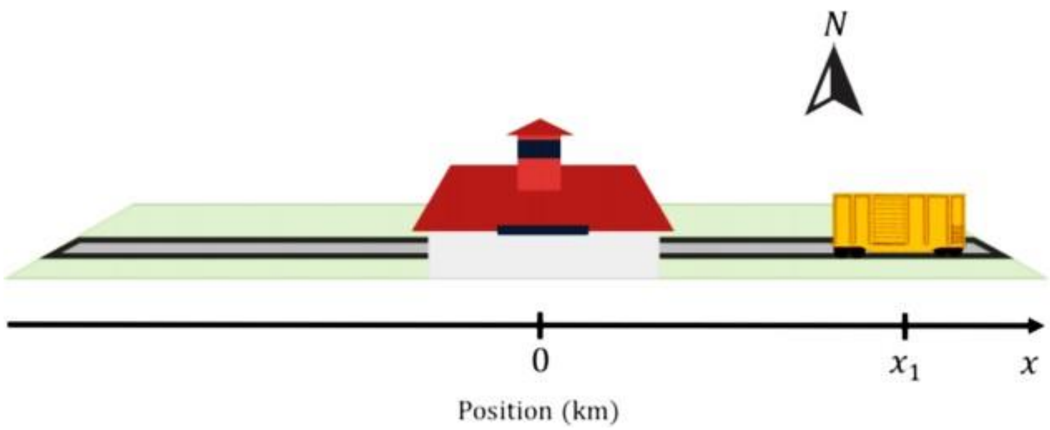


Figure (1): A 1D coordinate system describing the position of a train.

** In order to fully specify a one-dimensional coordinate system we need to choose:

- The location of the origin.
- The direction in which the coordinate, x, increases.
- The units in which we wish to express x.

In one dimension, it is common to use the variable x to define the position along the "x-axis". The x-axis is our coordinate system in one dimension.

3. 2D Coordinate systems

To describe the position of an object in two dimensions, we need to specify two numbers. The easiest way to do this is to define two axes, x and y. Figure (2) shows an example of such a coordinate system. The axes are perpendicular in "Cartesian" coordinate system.

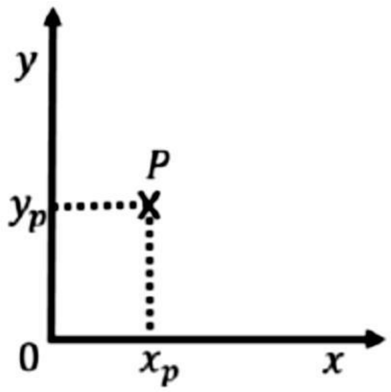


Figure (2): Example of Cartesian coordinate system and a point P with coordinates (x_p, y_p).

Another common choice is a "polar" coordinate system, where the position of an object is specified by a distance to the origin, r, and an angleθ, relative to a specified direction, as shown in Figure (3). Often, a polar coordinate system is defined alongside a Cartesian system, so that r is the distance to the origin of the Cartesian system and θ is the angle with respect to the axis.

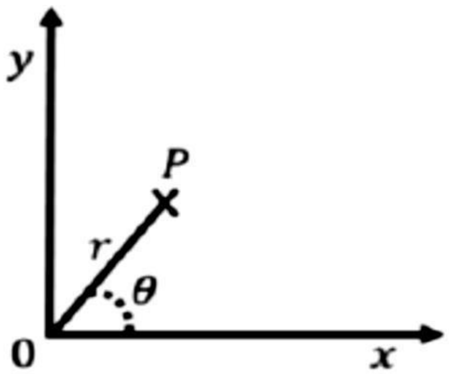


Figure (3): Example of a polar coordinate system and a point P with coordinates (r, θ) .

One can easily convert between the two Cartesian coordinates, x and y, and the two corresponding polar coordinates, r and θ:

$$
\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta) \\
r &= \sqrt{x^2+y^2} \\
\tan(\theta) &= \frac{y}{x}
\end{aligned}
$$

Polar coordinates are often used to describe the motion of an object moving around a circle, as this means that only one of the coordinates (θ) changes with time

4. 3D Coordinate systems

In three dimensions, we need to specify three numbers to describe the position of an object. In a three dimensional Cartesian coordinate system, we simply add a third axis, z, that is mutually perpendicular to both x and y. The position of an object can then be specified by using the three coordinates x, y, and z. Two additional coordinate systems are common in three dimensions: "cylindrical" and "spherical" coordinates. All three systems are illustrated in Figure (4) superimposed onto the Cartesian system.

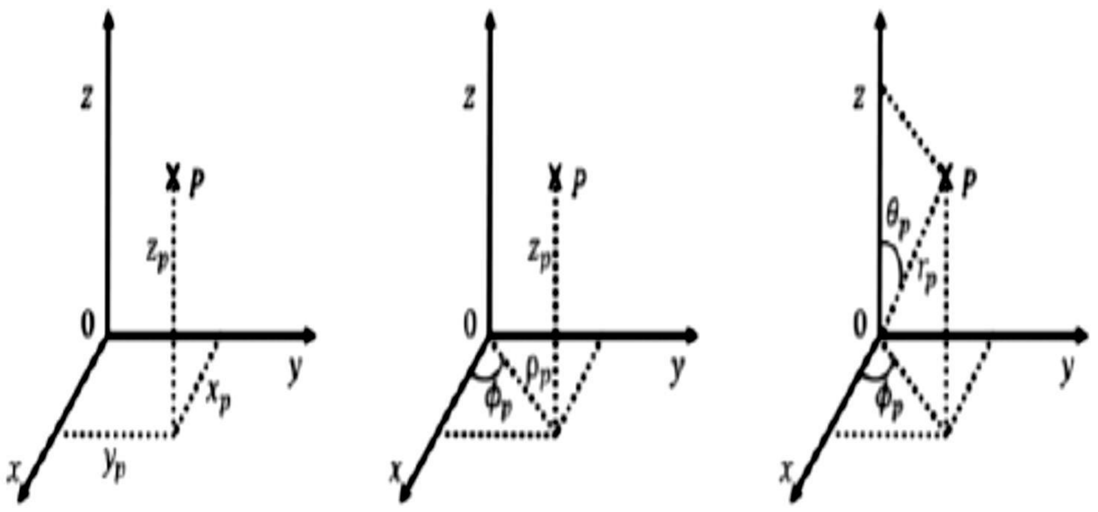


Figure (4): Cartesian (left), cylindrical (center) and spherical (right) coordinate systems used in three dimensions.

Cylindrical coordinates can be thought of as an extension of the polar coordinates. We keep the same Cartesian coordinate z to indicate the height above the x-y plane, however, we use the azimuthal angle,ϕ, and the radius,ρ, to describe the position of the projection of a point onto the x-y plane. ϕ is the angle between the x axis and the line from the origin to the projection of the point in the x-y plane and ρ is the distance between the point and the z axis.

**The cylindrical coordinates are related to the Cartesian coordinates by:

$$\rho = \sqrt{x^2 + y^2}$$

$$\tan(\phi) = \frac{y}{x}$$

$$z = z$$

In spherical coordinates, a point P is described by the radius, r, the polar angleθ , and the azimuthal angleϕ. The radius is the distance between the point and the origin. The polar angle is the angle with the z axis that is made by the line from the origin to the point. The azimuthal angle is defined in the same way as in polar coordinates.

**The spherical coordinates are related to the Cartesian coordinates by:

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\cos(\theta) = \frac{z}{r} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$
$$\tan(\phi) = \frac{y}{x}$$

5. Velocity and Acceleration in Plane Polar Coordinates

Let the polar coordinates r, θ to express the position of a particle moving in a plane. The position of the particle can be written as the product of the radial distance r by a unit radial vector e_r:

$$\mathbf{r} = r\mathbf{e}_r$$

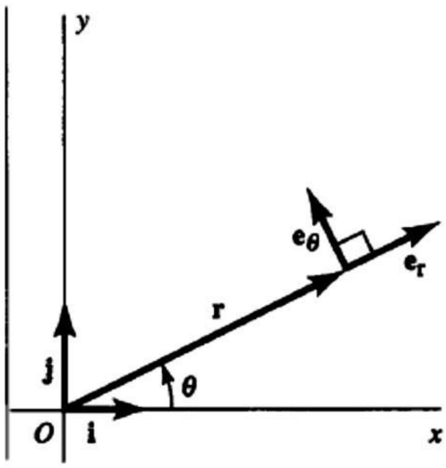


Figure (7):Unit vectors for plane polar coordinates.

As the particle moves, both r and e_r vary; thus, they are both functions of the time. Hence, if we differentiate with respect to t

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

By using Equation for the derivative of the unit radial vector, we can finally write the equation for the velocity as:

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

Thus, $\dot{r}$ is the radial component of the velocity vector, and $r\dot{\theta}$ is the transverse component.

The equation for the acceleration vector in plane polar coordinates.

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

The radial component of the acceleration vector is:

$$a_r = \ddot{r} - r\dot{\theta}^2$$

And the transverse component is:

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Example [*1*]:

A body moves in a spiral path in such a way that the radial distance decreases at a constant rate r = b – ct while the angular speed increases at a constant rate, $\dot{\theta} = kt$, Find the speed as a function of time.

By using equation of velocity

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

We have $\dot{r} = -c$ and $\ddot{r} = 0$.

$$\mathbf{v} = -c\mathbf{e}_r + (b - ct)kt\mathbf{e}_\theta$$

$$v = [c^2 + (b - ct)^2k^2t^2]^{1/2}$$

which is valid for $t \le b/c$. Note that $v = c$ both for $t = 0, r = b$ and for $t = b/c, r = 0$

Example **[2]:** A particle is moving along a spiral path with its polar coordinate position r = bt^2 and θ = ct where b and c is constant find the velocity and acceleration as a function of time.

By using equation of velocity

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\begin{aligned}\mathbf{v} &= \mathbf{e}_r \frac{d}{dt}(bt^2) + \mathbf{e}_\theta(bt^2)\frac{d}{dt}(ct) \\ &= (2bt)\mathbf{e}_r + (bct^2)\mathbf{e}_\theta\end{aligned}$$

By using the equation of acceleration

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta$$

$$\begin{aligned}\mathbf{a} &= \mathbf{e}_r(2b - bt^2c^2) + \mathbf{e}_\theta[0 + 2(2bt)c] \\ &= b(2 - t^2c^2)\mathbf{e}_r + 4bct\mathbf{e}_\theta\end{aligned}$$

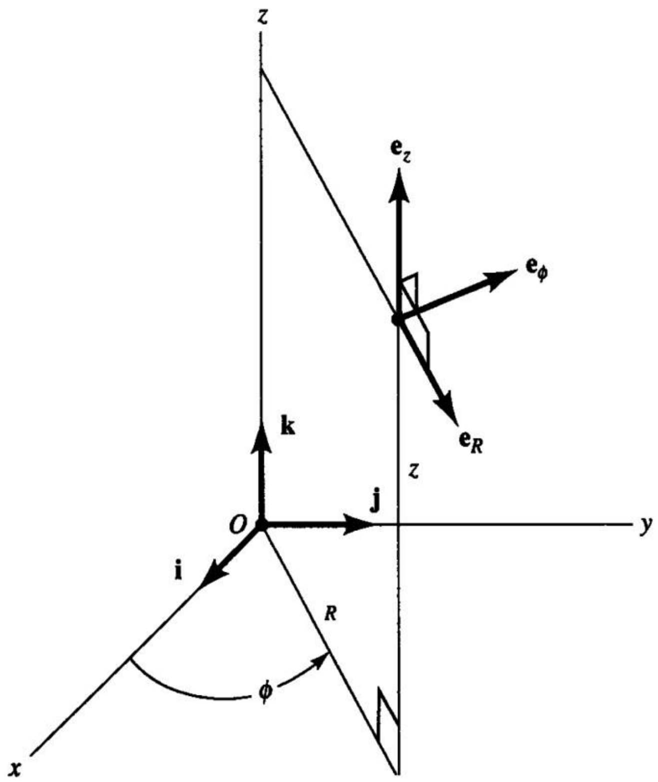
6. Velocity and Acceleration in Cylindrical Coordinates

In the case of three-dimensional motion, the position of a particle can be described in cylindrical coordinates R, ϕ, z. The position vector is then written as

$$\mathbf{r} = R\mathbf{e}_R + z\mathbf{e}_z$$

Where e_R is a unit radial vector in the x-y plane and e_z is the unit vector in the z direction.

A third unit vector e_ϕ is needed so that the three vectors e_R e_ϕ e_z constitute a right-handed triad,



The velocity and acceleration vectors are:

$$\mathbf{v} = \dot{R}\mathbf{e}_R + R\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z$$

$$\mathbf{a} = (\ddot{R} - R\dot{\phi}^2)\mathbf{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z$$

Example **[3]:**

A bead slides on a wire bent into the form of a helix, the motion of the bead being given in cylindrical coordinates by R = b, ϕ = wt, z = ct. Find the velocity and acceleration vectors as functions of time.

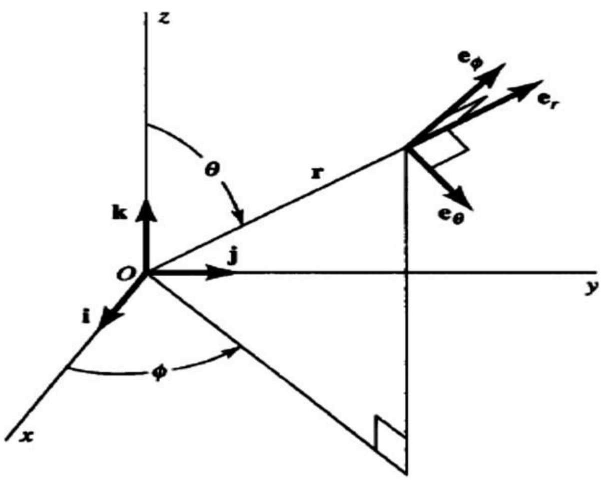
we find $\dot{R} = \ddot{R} = 0,\ \dot{\phi} = \omega,\ \ddot{\phi} = 0,\ \dot{z} = c,\ \ddot{z} = 0.$

By using the equation of velocity and acceleration

$$\mathbf{v} = b\omega\mathbf{e}_\phi + c\mathbf{e}_z$$
$$\mathbf{a} = -b\omega^2\mathbf{e}_R$$

7. Velocity and Acceleration in Spherical Coordinates

When spherical coordinates r,θ, ϕ are employed to describe the position of a particle, the position vector is written as the product of the radial distance r and the unit radial vector e_r, as with plane polar coordinates.



Thus, $$\mathbf{r} = r\mathbf{e}_r$$

The velocity vector in terms of its components in the rotated triad.

$$\mathbf{v} = \mathbf{e}_r\dot{r} + \mathbf{e}_\phi r\dot{\phi}\sin\theta + \mathbf{e}_\theta r\dot{\theta}$$

The acceleration vector in terms of its components in the triad

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi$$

Example **[2]:**

A wheel of radius b is placed in a gimbal mount and is made to rotate as follows. The wheel spins with constant angular speed $\omega_1$about its own axis, which in turn rotates with constant angular speed $\omega_2$about a vertical axis in such a way that the axis of the wheel stays in a horizontal plane and the center of the wheel is motionless. Use spherical coordinates to find the acceleration of any point on the rim of the wheel. In particular, find the acceleration of the highest point on the wheel.

Spherical coordinates r=b, $\theta=\omega_1 t$, $\phi= \omega_2$ t

Find the derivative $\dot{r} = \ddot{r} = 0,\ \dot{\theta} = \omega_1\ \ddot{\theta} = 0,\ \dot{\varphi} = \omega_2\ \ddot{\varphi} = 0$

Use the equation of acceleration we find

$$\mathbf{a} = (-b\omega_2^2 \sin^2\theta - b\omega_1^2)\mathbf{e}_r - b\omega_2^2 \sin\theta\cos\theta\,\mathbf{e}_\theta + 2b\omega_1\omega_2 \cos\theta\,\mathbf{e}_\phi$$

The point at the top has coordinate $\theta = 0$, so at that point

$$\mathbf{a} = -b\omega_1^2\mathbf{e}_r + 2b\omega_1\omega_2\mathbf{e}_\phi$$

The first term on the right is the centripetal acceleration, and the last term is a transverse acceleration normal to the plane of the wheel.