Analytic Mechanics

Sixth lecture

The work principle

M. Sc. Baraa Abd Alrda

M. Sc. *Heider Hassan Al-kaaby*

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Department of medical physics

Al-Mustaqbal University-College

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1.1 Velocity –Dependent Force

It often happens that the force acting on a particle is a function of the particle's velocity. This is true, for example, in the case of viscous resistance exerted on a body moving through a fluid. In the case of fluid resistance, it is found that, for low velocities, the resistance is approximately proportional to the velocity, whereas, for higher velocities, the resistance is more nearly proportional to the square of v. If there are no other forces acting, the differential equation of motion can be expressed as

$$F(v) = m\frac{dv}{dt}$$

A single integration yields t as a function of v

$$t = \int \frac{m\,dv}{F(v)} = t(v)$$

We can omit the constant of integration, since its value depends only on the choice of the time origin. Assuming that we can solve the above equation for v, namely,

$$v = v(t)$$

then a second integration gives the position x as a function of t

$$x = \int v(t)\,dt = x(t)$$

EXAMPLE

Suppose a block is projected with initial velocity v_0 on a smooth horizontal plane, but that there is air resistance proportional to v; that is, $F(v) = -cv$, where c is a constant of proportionality. (The x axis is along the direction of motion.) The differential equation of motion is

$$-cv = m\frac{dv}{dt}$$

which gives, upon integrating,

$$t = \int_{v_0}^{v} -\frac{m\,dv}{cv} = -\frac{m}{c}\ln\left(\frac{v}{v_0}\right)$$

We can easily solve for v as a function of t by multiplying by $-c/m$ and taking the exponent of both sides. The result is

$$v = v_0 e^{-ct/m}$$

Thus the velocity decreases exponentially with time. A second integration gives

$$x = \int_0^t v_0 e^{-ct/m}\,dt$$

$$= \frac{mv_0}{c}\left(1 - e^{-ct/m}\right)$$

We see, from the above equation, that the block never goes beyond the limiting distance mv_0/c.

1.2 Vertical Motion in a Resisting Medium. Terminal Velocity

An object falling vertically through the air or through any fluid is subject to viscous resistance. If the resistance is proportional to the first power of v (the linear case), we can express this force as $-cv$ regardless of the sign of v, because the resistance is always opposite to the direction of motion. The constant of proportionality c depends on the size and shape of the object and the viscosity of the fluid. Let us take the x axis to be positive upward. The differential equation of motion is then

$$-mg - cv = m\frac{dv}{dt}$$

If g is a constant, then we have a velocity-dependent force, and we can write

$$t = \int \frac{m\,dv}{F(v)} = \int_{v_0}^{v} \frac{m\,dv}{-mg - cv}$$
$$= -\frac{m}{c}\ln\frac{mg + cv}{mg + cv_0}$$

We can readily solve for v

$$v = -\frac{mg}{c} + \left(\frac{mg}{c} + v_0\right)e^{-ct/m}$$

The exponential term drops to a negligible value after a sufficient time ($t \gg m/c$), and the velocity approaches the limiting value $-mg/c$. The limiting velocity of a falling body is called the *terminal velocity;* it is that velocity at which the force of resistance is just equal and opposite to the weight of the body so that the total force is zero. The magnitude of the terminal velocity is called the *terminal speed.* The terminal speed of a falling raindrop, for instance, is roughly 10 to 20 ft per sec, depending on the size.

Equation (3.33) expresses v as a function of t, so a second integration will give x as a function of t:

$$x - x_0 = \int_0^t v(t)\,dt = -\frac{mg}{c}t + \left(\frac{m^2g}{c^2} + \frac{mv_0}{c}\right)(1 - e^{-ct/m})$$

Let us designate the terminal speed mg/c by v_t, and let us write τ (which we may call the *characteristic time*) for m/c. Equation (2.21) may then be written in the more significant form

$$v = -v_t + (v_t + v_0)e^{-t/\tau}$$

Thus, an object dropped from rest ($v_0 = 0$) will reach a speed of $1 - e^{-1}$ times the terminal speed in a time τ, $(1 - e^{-2})v_t$ in a time 2τ, and so on. After an interval of 10τ the speed is practically equal to the terminal value, namely $0.99995\ v_t$.

If the viscous resistance is proportional to v^2 (the quadratic case), the differential equation of motion is, remembering that we are taking the positive direction upward,

$$-mg \pm cv^2 = m\frac{dv}{dt}$$

The minus sign for the resistance term refers to upward motion (v positive), and the plus sign refers to downward motion (v negative). The double sign is necessary for any resistive force that involves an even power of v. As in the previous case, the differential equation of motion can be integrated to give t as a function of v:

$$t = \int \frac{m\,dv}{-mg - cv^2} = -\tau \tan^{-1}\frac{v}{v_t} + t_0 \qquad (rising)$$

$$t = \int \frac{m\,dv}{-mg + cv^2} = -\tau \tanh^{-1}\frac{v}{v_t} + t_0' \qquad (falling)$$

where

$$\sqrt{\frac{m}{cg}} = \tau \text{ (the characteristic time)}$$

and

$$\sqrt{\frac{mg}{c}} = v_t \text{ (the terminal speed)}$$

Solving for v,

$$v = v_t \tan\frac{t_0 - t}{\tau} \qquad (rising)$$

$$v = -v_t \tanh\frac{t - t_0'}{\tau} \qquad (falling)$$

If the body is released from rest at time $t = 0$, then $t_0' = 0$. We have then, from the definition of the hyperbolic tangent,

$$v = -v_t \tanh\frac{t}{\tau} = -v_t\left(\frac{e^{t/\tau} - e^{-t/\tau}}{e^{t/\tau} + e^{-t/\tau}}\right)$$

Again we see that the terminal speed is practically attained after the lapse of a few characteristic times, for example, for $t = 5\tau$, the speed is $0.99991\ v_t$. Graphs of speed versus time of fall for the linear and quadratic laws of resistance are shown in Figure 2.3. It is interesting to note that, in both the linear and the quadratic cases, the characteristic time τ is equal to v_t/g. For instance, if the terminal speed of a parachute is 4 ft per sec, the characteristic time is 4 ft per sec/32 ft per sec^2 = $\frac{1}{8}$ sec.