# *Analytic Mechanics*

*fifth lecture*

# Mass and force

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## 1. The concepts of kinetic and potential energy

It is generally true that the force that a particle experiences depends on the particle's position with respect to other bodies. This is the case, for example, with electrostatic and gravitational forces. It also applies to forces of elastic tension or compression. If the force is independent of velocity or time, then the differential equation for rectilinear motion is simply

$$F(x) = m\ddot{x}$$

It is usually possible to solve this type of differential equation by one of several methods. One useful and significant method of solution is to write the acceleration in the following way:

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx}$$

so the differential equation of motion may be written

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\frac{d(v^2)}{dx} = \frac{dT}{dx} \qquad 1$$

The quantity $T = \frac{1}{2}mv^2$ is called the *kinetic energy* of the particle. We can now express Equation (2.12) in integral form

$$\int F(x)\,dx = \int dT = \tfrac{1}{2}m\dot{x}^2 + \text{constant}$$

Now the integral $\int F(x)\,dx$ is the *work* done on the particle by the impressed force $F(x)$. Let us *define* a function $V(x)$ such that

$$-\frac{dV}{dx} = F(x) \qquad 2$$

The function $V(x)$ is called the *potential energy;* it is defined only to within an additive (arbitrary) constant. In terms of $V(x)$, the work integral is

$$\int F(x)\,dx = -\int \frac{dV}{dx}\,dx = -V(x) + \text{constant}$$

Consequently we may write

$$T + V = \tfrac{1}{2}mv^2 + V(x) = \text{constant} = E \qquad 3$$

We call $E$ the total energy. In words: For one-dimensional motion, if the impressed force is a function of position only, then the sum of the kinetic and potential energies remains constant throughout the motion. The force in this case is said to be *conservative*.[4] Nonconservative forces, that is, those for which no potential function exists, are usually of a dissipational nature, such as friction.

The motion of the particle can be obtained by solving the energy equation [Equation (2.14)] for $v$

$$v = \frac{dx}{dt} = \pm\sqrt{\frac{2}{m}[E - V(x)]} \qquad 4$$

which can be written in integral form

$$\int \frac{\pm dx}{\sqrt{\frac{2}{m}[E - V(x)]}} = t \qquad 5$$

thus giving $t$ as a function of $x$.

In view of Equation (3.21) we see that the expression for speed is real only for those values of $x$ such that $V(x)$ is less than or equal to the total

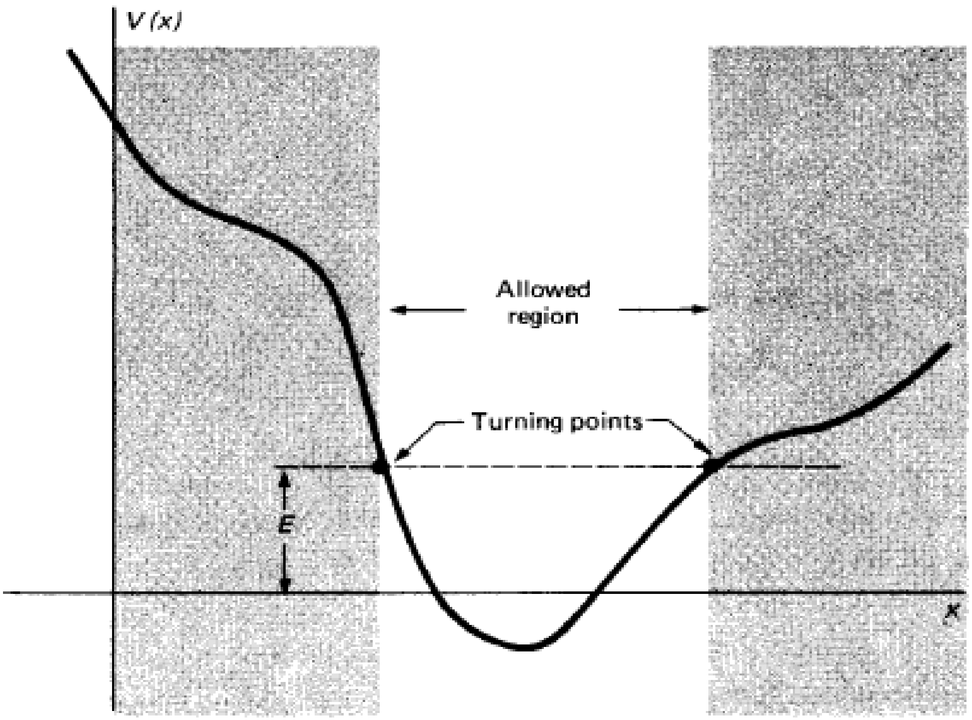


**FIGURE 1Graph of a potential energy function V(z) showing allowed region of motion and the turning points for a given value of the total energy E**

**A more complete discussion of conservative forces will be found in the next chapter. Physically, this means that the particle is confined to the region, or regions, for which the condition V(x) < E is satisfied. Furthermore, the speed goes to zero when V(x) = E. This means that the particle must come to rest and reverse its motion at those points for which the equality holds. These points are called the turning points of the motion. The above facts are illustrated in Figure 1.**

## EXAMPLE :

**The motion of a freely falling body discussed above under the case of a constant force is a special case of conservative motion. If we choose the z direction to be positive upward, then the gravitational force is equal to -mg, and the potential energy function is therefore given by V= mgx+C. Here C is an arbitrary constant whose value depends merely on the choice of the reference level for V. For C = 0, the total energy is just**

$$E = \tfrac{1}{2}m\dot{x}^2 + mgx$$

**Suppose, for example, that a body is projected upward with initial speed . Choosing r = 0 as the initial point of projection, we have**

$$E = \tfrac{1}{2}m{v_0}^2 = \tfrac{1}{2}m\dot{x}^2 + mgx$$

**The turning point is the maximum height attained by the body. It can be found by setting** $\dot{x} = 0.$ **Thus**

or

$$\tfrac{1}{2}mv_0^2 = mgx_{max}$$

$$h = x_{max} = \frac{v_0^2}{2g}$$

**The motion, as expressed by integrating the energy equation is given by**

$$\int_0^x (v_0^2 - 2gx)^{-1/2}\,dx = t$$

$$\frac{v_0}{g} - \frac{1}{g}(v_0^2 - 2gx)^{1/2} = t$$

**The student should verify that this reduces to the same relation between and t as that given by Equation** $x = \tfrac{1}{2}at^2 + v_0t + x_0$ **when a is set equal to -g.**

## 2- The Force as a Function of Time. The Concept of Impulse

**If the force acting on a particle is known explicitly as a function of time, then the equation of motion is**

$$F(t) = m\frac{dv}{dt}$$

**This can be integrated directly to give the linear momentum (and hence velocity) as a function of the time**

$$\int F(t)\, dt = mv(t) + C$$

**in which C is a constant of integration. The integral ( F(t) dt is called the impulse. It is equal to the momentum imparted to the particle by the force F(t).**

**The position of the particle as a function of time can be found by a second integration as follows**

$$x = \int v(t)\, dt = \int \left[ \int \frac{F(t')}{m}\, dt' \right] dt$$

**It should be noted that only in the case of the force being given as a function of t, is the solution of the equation of motion expressible as a simple double integral. In all other cases, the various methods of solving second order differential equations must be used to find the position x as a function of t.**

## EXAMPLE

**A block is initially at rest on a smooth horizontal surface. At time t = 0 a constantly increasing horizontal force is applied: F = ct. Find the velocity and the displacement as functions of time.**

**We have, for the differential equation of motion,**

$$ct = m\frac{dv}{dt}$$

$$v = \frac{1}{m}\int_0^t ct\,dt = \frac{ct^2}{2m}$$

$$x = \int_0^t \frac{ct^2}{2m}\,dt = \frac{ct^3}{6m}$$

where the initial position of the block is at the origin ($x = 0$).