Analytic Mechanics

Second lecture

Vector calculus and Kinematic of a particle

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1. Derivative of vector

Consider a vector $\vec{A}$ whose components are functions of a single variable u. The vector may represent position, velocity, and so on. The parameter u is usually the time t, but it can be any quantity that determines the components of $\vec{A}$

$$\vec{A}(u) = \hat{i}A_x(u) + \hat{j}A_y(u) + \hat{k}A_z(u)$$

Derivative of a vector is a vector whose components are ordinary derivatives

$$\frac{d\vec{A}}{du} = \hat{i}\frac{dA_x}{du} + \hat{j}\frac{dA_y}{du} + \hat{k}\frac{dA_z}{du}$$

**The derivative of the sum of two vectors is equal to the sum of the derivatives,

$$\frac{d}{du}(\vec{A} + \vec{B}) = \frac{d\vec{A}}{du} + \frac{d\vec{B}}{du}$$

** Derivative of products of vectors

$$\frac{d(n\vec{A})}{du} = \frac{dn}{du}\vec{A} + n\frac{d\vec{A}}{du}$$

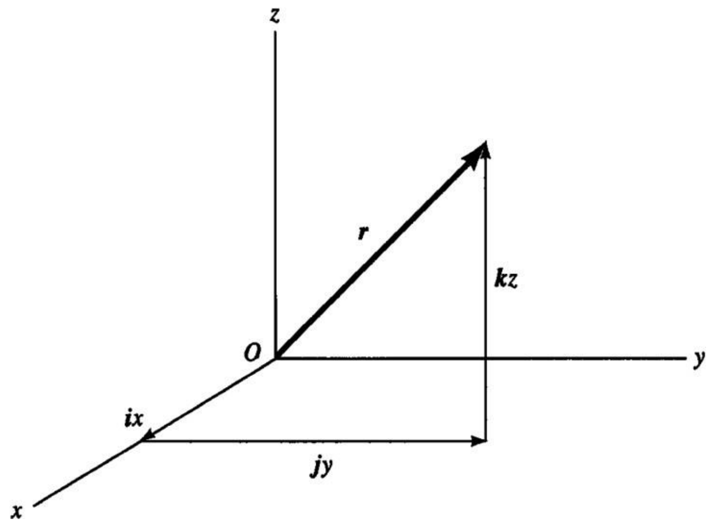
$$\frac{d(\vec{A}\cdot\vec{B})}{du} = \frac{d\vec{A}}{du}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{du}$$

$$\frac{d(\vec{A}\times\vec{B})}{du} = \frac{d\vec{A}}{du}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{du}$$

2. Position Vector of a Particle: Velocity and Acceleration

The position of a particle can be specified by a single vector, the displacement of the particle relative to the origin of the coordinate system. This vector is called the position vector of the particle.

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$



** The velocity vector.

If the position vector for a particle $\vec{r}$ and the parameter is the lime t the derivative of r with respect to t is called the velocity, which we shall denote by v:

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}$$

Where the dots indicate differentiation with respect to t. The magnitude of the velocity is called the speed. In rectangular components the speed is just

$$v = |\vec{v}| = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{\frac{1}{2}}$$

**The acceleration vector

The time derivative of the velocity is called the acceleration. Denoting the acceleration with a,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

In rectangular components:

$$\vec{a} = \hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

Example: Projectile Motion

Let us examine the motion represented by the equation:

$$\vec{r}(t) = \hat{i}bt + \hat{j}(ct - \frac{gt^2}{2}) + \hat{k}0$$

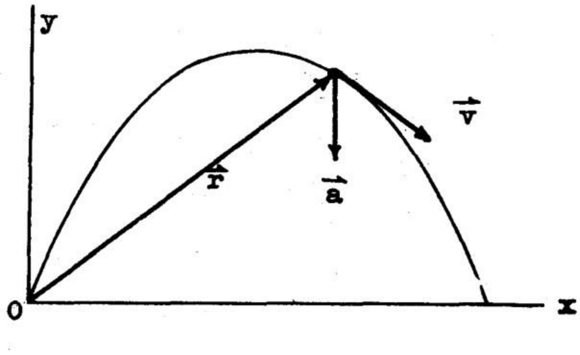
This represents motion in the x-y plane, because the z component is constant and equal to zero. The velocity v is obtained by differentiating with respect to t, namely:

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}b + \hat{j}(c - gt)$$

The acceleration, likewise, is given by:

$$\vec{a} = \frac{d\vec{v}}{dt} = -\hat{j}g$$

Thus, a is in the negative y direction and has the constant magnitude g.
The path of motion is a parabola, as shown in Figure.



The speed v varies with t according to the equation:

$$v = \left[b^2 + (c - gt)^2\right]^{\frac{1}{2}}$$

Example: Circular Motion

Suppose the position vector of a particle is given by:

$$\vec{r} = \hat{i}b\sin\omega t + \hat{j}b\cos\omega t + \hat{k}c$$

Where a, is a constant.

Let us analyze the motion. The distance from the origin remains constant:

$$|\vec{r}| = r = (b^2\sin^2\omega t + b^2\cos^2\omega t + c^2)^{\frac{1}{2}}$$
$$= (b^2 + c^2)^{\frac{1}{2}}$$

Differentiating r, we find the velocity vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}b\omega\cos\omega t - \hat{j}b\omega\sin\omega t + \hat{k}0$$

The velocity vector is parallel to the x-y Plane. The particle moves with constant speed:

$$v = |\vec{v}| = (b^2\omega^2\cos^2\omega t + b^2\omega^2\sin^2\omega t)^{\frac{1}{2}} = b\omega$$

The acceleration is:

$$\vec{a} = \frac{d\vec{v}}{dt} = -\hat{i}b\omega^2 \sin\omega t - \hat{j}b\omega^2 \cos\omega t$$

In this case the acceleration is perpendicular to the velocity, because the dot product of v and a vanishes:

$$\vec{v}.\vec{a} = (b\omega \cos\omega t)(-b\omega^2 \sin\omega t)+(-b\omega \sin\omega t)(-b\omega^2 \cos\omega t)=0$$

3.Vector integration

Suppose that the time derivative of a vector **r** is given in rectangular coordinates where each component is known as a function of time, namely,

$$\frac{d\vec{r}}{dt} = \hat{i}f_1(t) + \hat{j}f_2(t) + \hat{k}f_3(t)$$

It is possible to integrate with respect to t to obtain

$$\mathbf{r} = \mathbf{i}\int f_1(t)\,dt + \mathbf{j}\int f_2(t)\,dt + \mathbf{k}\int f_3(t)\,dt$$

Example:

The velocity vector of a moving particle is given by:

$$\vec{v} = \hat{i}A + \hat{j}Bt + \hat{k}Ct^{-1}$$

in which A,B,C are constants. Find r

$$\vec{r} = \hat{i}\int A\,dt + \hat{j}\int Bt\,dt + \hat{k}\int Ct^{-1}dt$$
$$= \hat{i}At + \hat{j}B\frac{t^2}{2} + \hat{k}C\ln t + \vec{r}_0$$

Where r_o is the constant of integration.