Quantum Mechanics

Fifth Lecture

Mathematical Representation of W.F

Dr. Nasma Adnan

Third Stage

Department of Medical Physics

Al-Mustaqbal University-College

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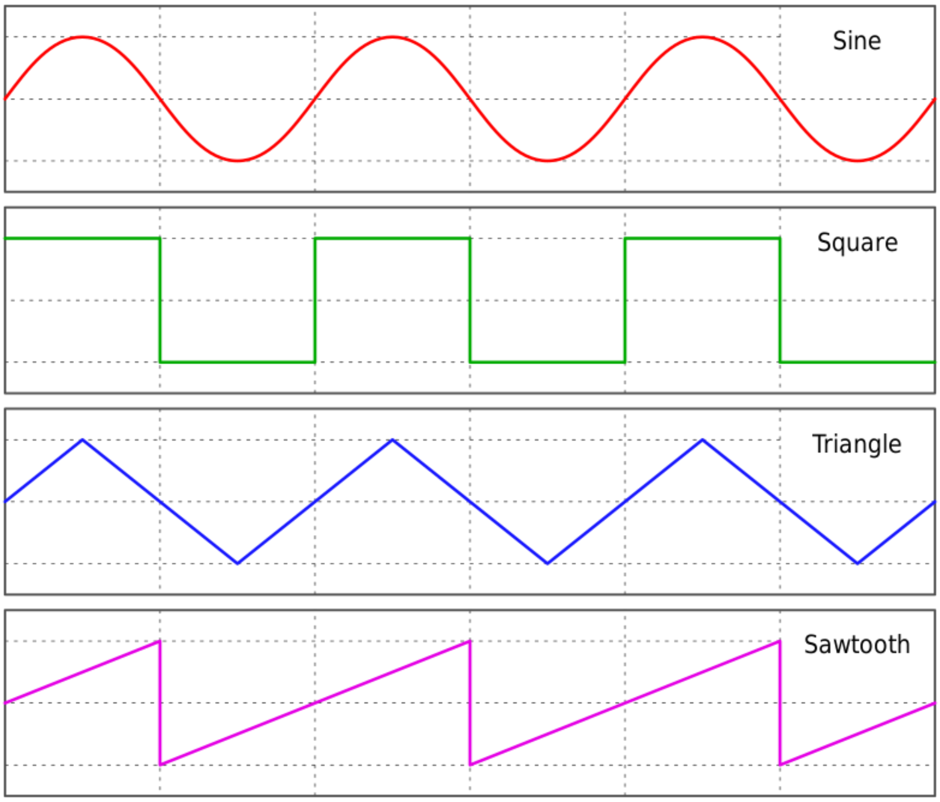
Mathematically, waves can be expressed as:

- Sinusoidal waves
- Square waves
- Triangle waves
- Saw-tooth wave



What is sinusoidal waves?

The sine or sinusoidal wave ***is a curve that describes repetitive oscillation. It is the wave form in which the amplitude is always proportional to sine of its displacement angle at every point of time.***

- Sinusoidal travelling wave with definite wavelength λ and period τ, or equivalently definite wave number, $k = 2\pi\backslash\lambda$, and angular frequency, $\omega = 2\pi\backslash\tau$. Such a wave may be represented by the mathematical function

$$\Psi(x, t) = A\cos(kx - \omega t)$$

- where A is a constant.
- At each point x, the function Ψ (x, t) oscillates with amplitude A and period 2π / ω.
- At each time t, the function Ψ (x, t) undulates with amplitude A and wavelength 2π =*k*.
- Moreover, these undulations move like vertically polarized, transverse, travelling wave.
- In the direction of increasing x with velocity ω/k; for example,
  - ✓ The maximum of $\Psi(x, t)$ corresponding to $kx - \omega t = 0$ occurs at the position $x = \omega t/k$.
  - ✓ The minimum corresponding to $kx - \omega t = \pi$ occurs at the position x = λ/2 + ωt/*k*
  - ✓ In both cases the position moves with velocity ω /*k*

- The function sin ($kx - \omega t$), like cos ($kx - \omega t$), also represents a sinusoidal travelling wave with wave number *k* and angular frequency ω. Because

$$\sin(kx - \omega t) = \cos(kx - \omega t - \pi/2)$$

- The undulations and oscillations of sin ($kx - \omega t$), are out of step with those of cos ($kx - \omega t$), the waves sin ($kx - \omega t$), and cos ($kx - \omega t$) are said to have a phase difference of π/2. The most general sinusoidal travelling wave with wave number k and angular frequency ω is the linear superposition:

$$\Psi(x, t) = A\cos(kx - \omega t) + B\sin(kx - \omega t)$$

  - ✓ where A and B are arbitrary constants.
- Sinusoidal travelling waves are represented by ***complex exponential functions*** of the form

$$\Psi(x, t) = A\,e^{i(kx - \omega t)}$$

- Complex exponential function consists of two parts: real and imaginary part

$$e^{i(kx-\omega t)} = \cos(kx-\omega t) + i\sin(kx-\omega t)$$

- a complex exponential provides a natural description of a ***de Broglie wave***.

Linear superposition's of sinusoidal waves

- ***Two sinusoidal waves moving in opposite directions may be combined to form standing waves***.
- ***For example***, the linear superposition:

$$A\cos(kx-\omega t) + A\cos(kx+\omega t)$$

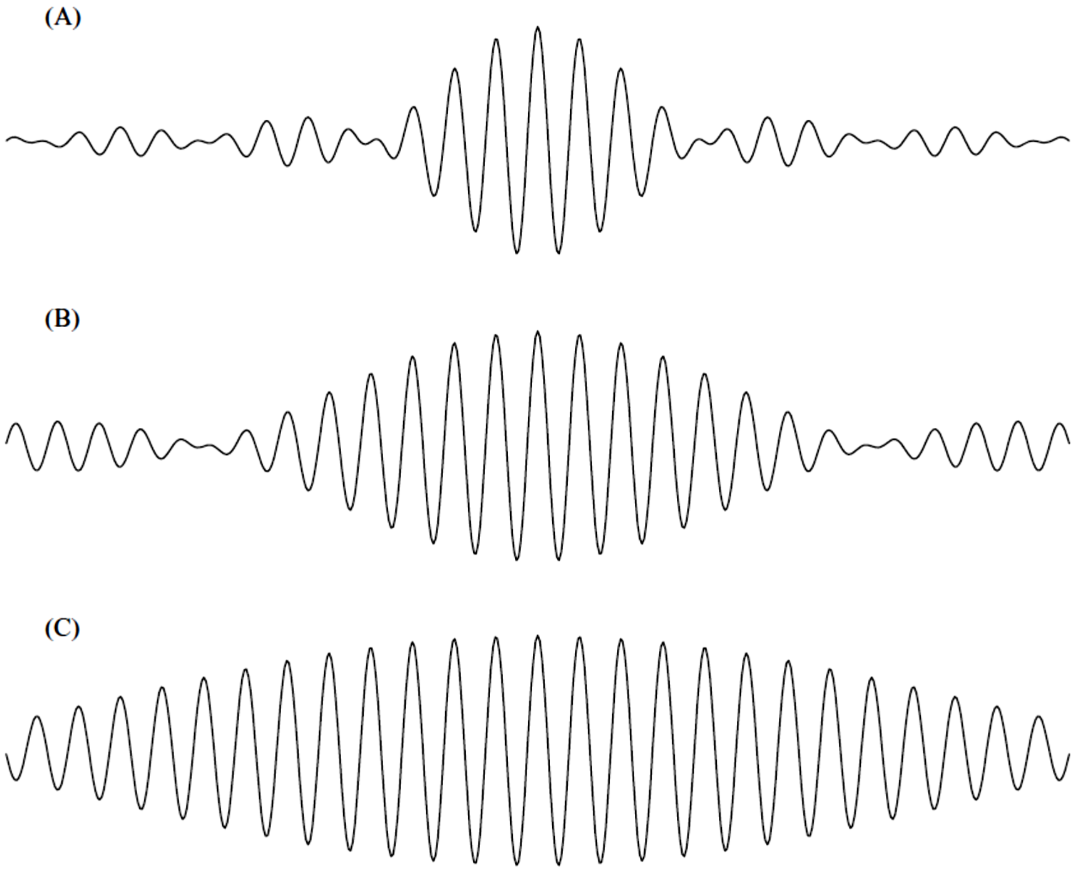
- gives rise to the wave ***2A cos kx cos ωt***. This wave oscillates with period ***2π\ω*** and undulates with wavelength ***2π=k***, but these oscillations and undulations do not propagate.
- Alternatively, many sinusoidal waves may be combined to form a wave packet.
- For example, the mathematical form of a wave packet formed by a linear superposition of sinusoidal waves with constant amplitude A and wave numbers in the range ***k - Δk to k + Δk***

$$\Psi(x,t) = \int_{k-\Delta k}^{k+\Delta k} A\cos(k'x-\omega' t)\, dk'.$$

- If k is positive, this wave packet travels in the positive x direction, and in the negative x direction if k is negative.
- ***The initial shape of the wave packet***, i.e. the shape at t = 0, may be obtained by evaluating the integral

$$\Psi(x,0) = \int_{k-\Delta k}^{k+\Delta k} A\cos k'x\, dk'$$

$$\Psi(x,0) = S(x)\cos kx, \qquad \text{where} \quad S(x) = 2A\Delta k\,\frac{\sin(\Delta k x)}{(\Delta k x)}$$



- The three diagrams show how the length of a wave packet increases as the range of wave numbers Δk decreases.
- The value of AΔk is constant.
- Δk equals k=8 in diagram (A), Δk equals k=16 in diagram (B) and Δk equals k=32 in diagram (C).
- In general, the length of a wave packet is inversely proportional to Δk and becomes infinite in extent as Δk → 0.
- The velocity of propagation of a wave packet, and the possible change of
- shape as it propagates, ***depend on the relation between the angular frequency and wave number.***
- This relation, the function ω(k), is called the ***dispersion relation*** because it determines whether the waves are dispersive or non-dispersive.

Dispersive and non-dispersive waves

- The most familiar example of a non-dispersive wave is an ***electromagnetic wave in the vacuum.***
- A non-dispersive wave has a dispersion relation of the form $\boldsymbol{\omega = ck}$, where c is a constant so that the velocity of a sinusoidal wave, $\boldsymbol{\omega \backslash k = c}$, is independent of the wave number k.
- A wave packet formed from a linear superposition of such sinusoidal waves travels without change of shape because each sinusoidal component has the same velocity.
- Non-dispersive waves are governed by a partial differential equation called the classical wave equation. For waves travelling in three dimensions, it has the form:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- for waves travelling in one dimension, the x direction says, it has the form

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

- The dispersion relation is more complicated than $\omega = ck$ so that the velocity of propagation of a sinusoidal wave, $\omega \backslash k$, depends upon the wave number k.
- Hence a packet of dispersive waves will, in general, change shape as it propagates.
- However, if the packet is composed of waves with a narrow range of wave numbers, it has a well-defined velocity of propagation. This velocity is called the group velocity and it is given by

$$\upsilon_{group} = \frac{d\omega}{dk}$$

- whereas the velocity of a simple sinusoidal wave, ω \ k, is called the phase velocity.
- ***The group velocity* describes the motion of a localized disturbance due to constructive interference of many sinusoidal waves.**
- If we have constructive interference of two sinusoidal waves with wave numbers k_1 and k_2 and angular frequencies ω_1 and ω_2 which is formed when the waves are in phase; i.e. when

$$k_1 x - \omega_1 t = k_2 x - \omega_2 t$$

- By rearranging this equation, we find that the position of this point of constructive interference is given by:

$$x = \left(\frac{\omega_1 - \omega_2}{k_1 - k_2}\right) t$$

❖ *What is group velocity for water wave?*

- From the dispersion relation

$$\omega = \sqrt{gk}$$

- where g is the acceleration due to gravity. The velocity of a sinusoidal water wave, the so-called phase velocity, is

$$\upsilon_{phase} = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \qquad \upsilon_{group} = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}}.$$

- Thus, for water waves, the group velocity is exactly one-half of the phase velocity. In other words, the sinusoidal waves forming the packet, travel at twice the speed of the region of maximum disturbance formed by the interference of these waves. However, the shape of the disturbance will change as it propagates; in general, it will tend to spread out.