Lecture No. 4

Torsion of Circular Shafts

4.1 Introduction

In many engineering applications, members are required to carry torsional loads. In this lecture, we consider the torsion of circular shafts. Because a circular cross section is an efficient shape for resisting torsional loads, circular shafts are commonly used to transmit power in rotating machinery. Derivation of the equations used in the analysis follows these steps:

Make simplifying assumptions about the deformation based on experimental evidence.

Determine the strains that are geometrically compatible with the assumed deformations.

Use Hooke's law to express the equations of compatibility in terms of stresses.

Derive the equations of equilibrium. (These equations provide the relationships between the stresses and the applied loads.)

4.2 Torsion of Circular Shafts

Consider the solid circular shaft, shown in the Figure 2.1, and subjected to a torque T at the end of the shaft. The fiber AB on the outside surface, which is originally straight, will be twisted into a helix AB′ as the shaft is twist through the angle θ. During the deformation, the cross sections remain circular (NOT distorted in any manner) - they remain plane, and the radius r does not change.

Besides, the length L of the shaft remains constant. Based on these observations, the following assumptions are made:

The material is homogeneous, i.e. of uniform elastic properties throughout.

The material is elastic, following Hooke's law with shear stress proportional to shear strain.

The stress does not exceed the elastic limit or limit of proportionality.

Circular cross sections remain plane (do not warp) and perpendicular to the axis of the shaft.

Cross sections do not deform (there is no strain in the plane of the cross section).

The distances between cross sections do not change (the axial normal strain is zero).

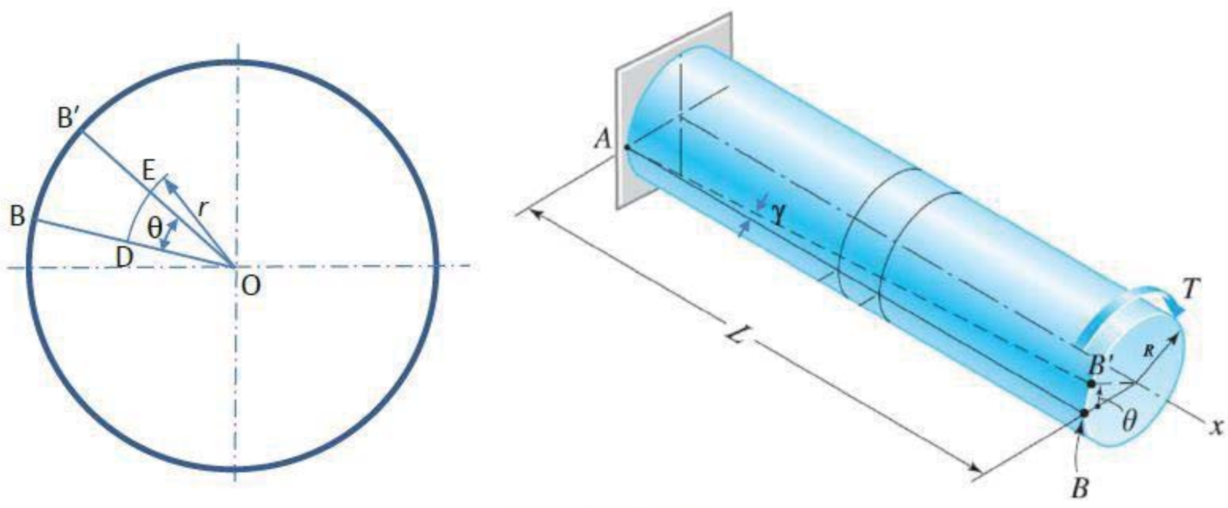


Figure 5.1: Deformation of a circular shaft caused by the torque T.

$\delta s = DE = r\theta$ ………………………………………………………….. (5.1)

Where the subscript s denotes shear, r is the distance from the origin to any interested fiber, and θ is the angle of twist.

From Figure 5.1

$\gamma L = r\theta$

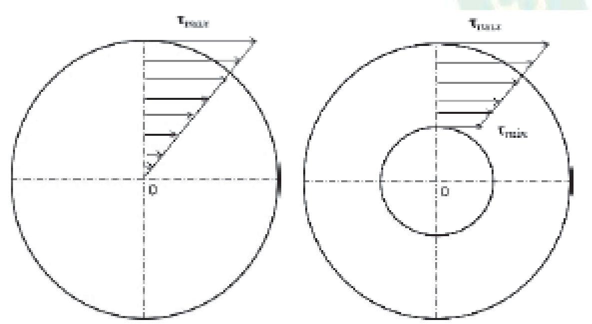
The unit deformation of this fiber is,

$$\gamma = \frac{\delta s}{L} = \frac{r\theta}{L} \quad \text{...} \quad (5.2)$$

Shear stress can be determined using Hooke's law as:

$$\tau = \frac{G}{\gamma} = \frac{r\theta}{L} G$$

Note: since, $\tau = \frac{r\theta}{L} G$ =const. therefore, the conclusion is that the shearing stress at any internal fiber varies linearly with the radial distance from the axis of the shaft.



For the shaft to be in equilibrium, the resultant of the shear stress acting on a cross section must be equal to the internal torque T acting on that cross section. Figure 2.2 shows a cross section of the shaft containing a differential element of area dA located at the radial distance r from the axis of the shaft. The shear force acting on this area is $dF = \tau\, dA$, directed perpendicular to the radius. Hence, the torque of dF about the center O is:

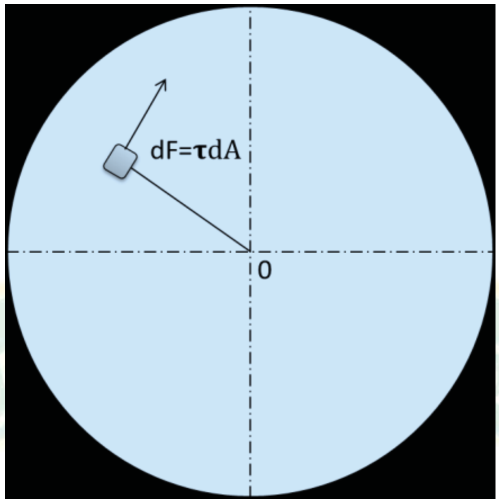


Figure 5.2: The resultant of the shear stress acting on the cross section.

$T=\int r dF = \int r\, \tau\, dA$ ……………………………………….(5.4)

Substituting equation (5.3) into equation (5.4),

$Tr=\int r(\frac{G\theta}{L}) r dA = \frac{G\theta}{L}\int r^2 dA$

Since $\int r^2 dA = J$, the polar 2nd moment of area (or polar moment of inertia) of the cross section

$T=\frac{G\theta}{L} J$

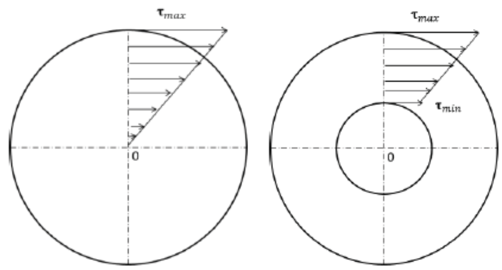
Rearranging the above equation,

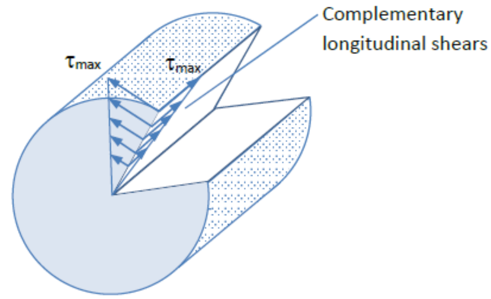
$\theta= \frac{TL}{JG}$…………………………………………………….(5.5)

where T is the applied torque (N.m), L is length of the shaft (m), G is the shear modulus (N/$m2$), J is the polar moment of inertia ($m4$), and θ is the angle of twist in radians.

From equations (5.5) and (5.3),

$\tau=(\frac{G\theta}{L})\ r=\frac{T\,J}{r}$





4.3 Polar Moment of Inertia

- Solid Shaft

Consider the solid shaft shown, therefore,

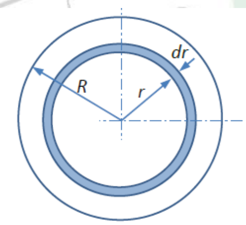
$$J = \int r^2 dA = \int_0^R r^2 (2\pi r dr) = 2\pi \int_0^R r^3 dr$$

which yields,

$$J = 2\pi [\frac{r^4}{4}]_0^R = \frac{\pi}{2} R^4$$

or

$$J = \frac{\pi d^4}{32}$$



- Hollow Shaft

The above procedure can be used for calculating the polar moment of inertia of the hollow shaft of inner radius Ri and outer radius Ro,

$$J = 2\pi \int_{R_i}^{R_o} r^3 dr = \frac{\pi}{2}(R_o^4 - R_i^4)$$

or

$$J = \frac{\pi}{32}(D_o^4 - D_i^4)$$

The maximum shear stress is found (at the surface of the shaft) by replacing r by the radius R, for solid shaft, or by , for the hollow shaft, as

$$\tau_{max} = \frac{2T}{\pi R^3} = \frac{16T}{\pi D^3} \quad \rightarrow solid\ shaft$$

$$\tau_{max} = \frac{2TR}{\pi(R^4 - r^4)} = \frac{16TD_o}{\pi\left(D_o^4 - D_i^4\right)} \quad \rightarrow hollow\ shaft$$

4.4 Composite Shafts –

Series Connection

If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is therefore termed series-connected, as shown in Figure 5. 4. In such cases the composite shaft strength is treated by considering each component shaft separately, applying the torsion theory to each in turn; the composite shaft will therefore be as weak as its weakest component. If relative dimensions of the various parts are required then a solution is usually effected by equating the torques in each shaft, e.g. for two shafts in series

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2}$$

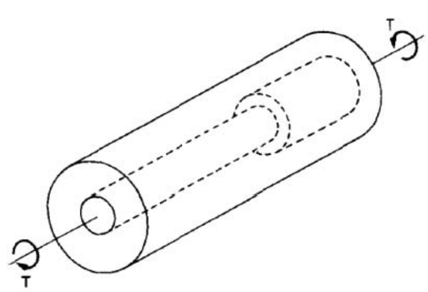


Figure 5.4: “Series connected” shaft - common torque

Composite Shafts - Parallel Connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel (Figure 2.5).

For parallel connection,

Total Torque $T = T_1 + T_2$…………………………………………………………… (5.7)

In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2}$$

or

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2}\left(\frac{L_2}{L_1}\right)$$

Thus two equations are obtained in terms of the torques in each part of the composite shaft and these torques can therefore be determined.

In case of equal lengths,becomes

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2}$$

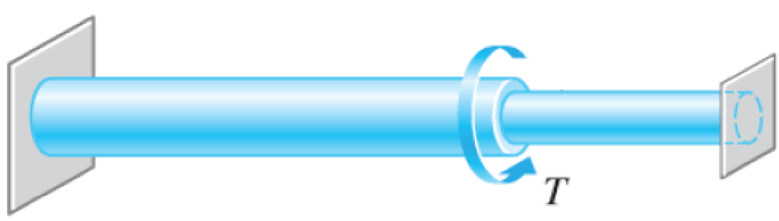


Figure 5.5: “Parallel connected” shaft - shared torque.

4.5 Power Transmitted by Shafts

If a shaft carries a torque T Newton meters and rotates at θ rad/s it will do work at the rate of

$T\omega$ Nm/s (or joule/s).

Now the rate at which a system works is defined as its power, the basic unit of power being the Watt (1 Watt = 1 Nm/s).

Thus, the power transmitted by the shaft:

= $T\omega$ Watts.

Since the Watt is a very small unit of power in engineering terms use is normally made of S.I. multiples, i.e. kilowatts (kW) or megawatts (MW).

Example1: A solid shaft in a rolling mill transmits 20 kW at 120 r.p.m. Determine the diameter of the shaft if the shearing stress is not to exceed 40MPa and the angle of twist is limited to 6° in a length of 3m. Use G=83GPa.

Solution/

$P = 20\,kW$; $N = 120\,rpm$

$d = ?$; $\tau = 40\,MPa$

$\theta = 6^\circ$; $L = 3\,m$

$G = 83\,GPa$

* $P = T * \omega$

$$\Rightarrow T = \frac{20*10^3}{\frac{2\pi N}{60}} = \frac{20*10^3}{\frac{2\pi*120}{60}} = 1591.5\ N.m$$

* القانون العام للالتواء

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$$

Based on τ /

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\Rightarrow \frac{1591.5*10^3}{\frac{\pi}{32} d^3} = \frac{40}{\frac{d}{2}}$$

[illegible]

$d = 58.73\ mm$

Based on θ /

$$\frac{T}{J} = \frac{G\theta}{L}$$

$$\frac{1591.5}{\frac{\pi}{32} d^4} = \frac{83*10^3 * 6 * \frac{\pi}{180}}{3*10^3}$$

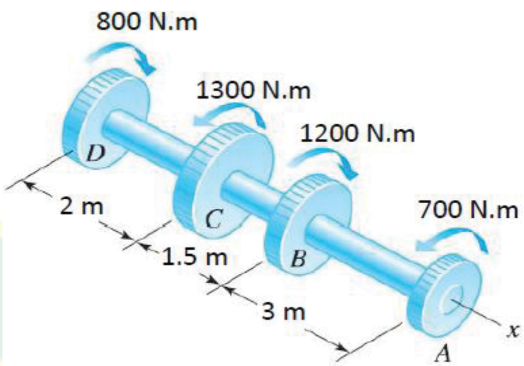
[illegible]

$d = 48.6\ mm$

The safe diameter is

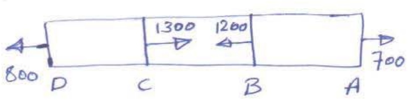
$d = 58.73\ mm$ → Ans.

Example 2 A steel shaft with constant diameter of 50 mm is loaded as shown in the figure by torques applied to gears fastened to it. Using G= 83 GPa, compute in degrees the relative angle of rotation between gears A and D.



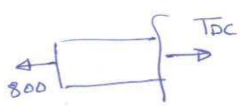
Solution/

* باستخدام قاعدة لكف اليمنى (نقوم بتمثيل ال Torque ك vector في Free body diag.



* باستخدام طريقة ال superposition نجد ال Torque في كل مقطع.

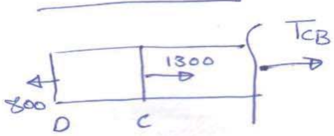
Section ①



$\Sigma T = 0$

$T_{DC} = 800$ N.m . C.CW

Section ②

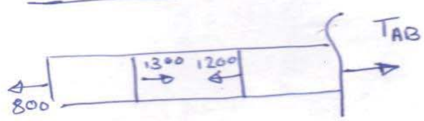


$\Sigma T = 0$

$-800 + 1300 + T_{CB} = 0$

$T_{CB} = -500$ N.m (C.W)

Section 3



$\Sigma T = 0$

$-800 + 1300 - 1200 + T_{AB} = 0$

$T_{AB} = 700$ N.m C.CW

$\frac{T}{J} = \frac{G\theta}{L} \Rightarrow \theta = \frac{TL}{GJ}$

$\theta_t = \theta_{AB} + \theta_{BC} + \theta_{CD}$

$= \frac{T_{AB} L_{AB}}{GJ} + \frac{T_{BC} L_{BC}}{GJ} + \frac{T_{CD} L_{CD}}{GJ}$

$= \frac{1}{GJ}\left[T_{AB} L_{AB} + T_{BC} L_{BC} + T_{CD} L_{CD}\right]$

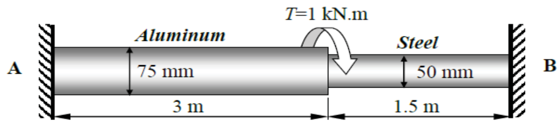
$\theta_{A/D} = \frac{1}{83 \times 10^3 \times \frac{\pi}{32}(50)^4}\left[700 \times 10^3 \times 3000 - 500 \times 10^3 \times 1500 + 800 \times 10^3 \times 2000\right]$

$= 5.79 \times 10^{-2}$ rad

$= 5.79 \times 10^{-2} \times \frac{180}{\pi}$

$\theta_{A/D} = 3.32^\circ \longrightarrow$ Ans.

Example 3 A compound shaft made of two segments: solid steel and solid aluminum circular shafts. The compound shaft is built-in at A and B as shown in the figure. Compute the maximum shearing stress in each shaft. Given Gal=28GPa, Gst = 83 GPa.



في هذا السؤال اعطانا ربط توازي (Parallel) وطلب منا (τ)
لكل من الالمنيوم والفولاذ (steel)

$T_T = 1000 * 10^3$ N.mm

$d_{al} = 75$ mm ; $d_s = 50$ mm

$L_{al} = 3000$ mm ; $L_s = 1500$ mm

$G_{al} = 28 * 10^3$ MPa ; $G_s = 83 * 10^3$ MPa

* For parallel connection

* $T_T = T_{al} + T_s$

$1000 * 10^3 = T_{al} + T_s$ —(1)

* $\theta_{al} = \theta_{st}$

$\frac{T}{J} = \frac{G\theta}{L} \Rightarrow \theta = \frac{TL}{GJ}$

$$\frac{T_{al} L_{al}}{J_{al} G_{al}} = \frac{T_{st} L_{st}}{J_{st} G_{st}}$$

$$\frac{T_{al} * 3000}{\frac{\pi}{32}(75)^4 * 28 * 10^3} = \frac{T_{st} * 1500}{\frac{\pi}{32}(50)^4 * 83 * 10^3}$$

$T_{al} = 0.854\, T_s$ —(2)

sub (2) in (1)

$1000 * 10^3 = 0.854\, T_s + T_s$

$T_s = 539.39 * 10^3$ N.mm

sub in (2)

$T_{al} = 0.854(539.39)$

$T_{al} = 460.64 * 10^3$ N.mm

* $\frac{T}{J} = \frac{\tau}{R} \Rightarrow \tau = \frac{TR}{J}$

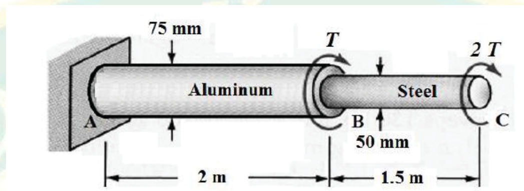
$$\tau_{al} = \frac{460.64 * 10^3 * \frac{75}{2}}{\frac{\pi}{32}(75)^4}$$

$\tau_a = 5.57$ MPa → Ans

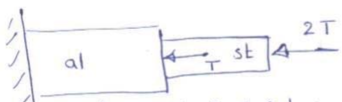
$$\tau_{st} = \frac{539.39 * \frac{50}{2} * 10^3}{\frac{\pi}{32}(50)^4}$$

$\tau_{st} = 22$ MPa → Ans

Example 4 A compound shaft consisting of an aluminum segment and a steel is acted upon by two torque as shown in the figure. Determine the maximum permissible value of T subjected to the following : $\tau s \le 100$MPa,$a \le 70$MPa , and the angle of rotation of the free end limited to 12°. Use Gs=83GPa and Ga=28 GPa.



ملاحظة/ هذا السؤال لا تنطبق عليه شروط ربط التوالي، لذلك نقوم بتطبيق طريقة الـ (superposition) على كل جزء من الـ (shaft) وحل السؤال بالاستناد الى (T) & (θ)

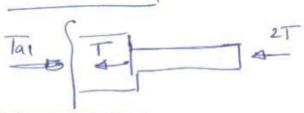


بتطبيق قاعدة الكف اليمنى

Section ①

$T_{st} = 2T$

Section ②



$T_{al} = 3T$

* Based on τ

$\frac{T}{J} = \frac{\tau}{R} \Rightarrow T = \frac{\tau J}{R}$

$\Rightarrow T_{al} = \frac{\tau_{al} J_{al}}{R_a}$

$3T = \frac{70 * \frac{\pi}{32} * (75)^4}{\frac{75}{2}}$

$T = 1932815.793$ N.mm

$T_{st} = \frac{\tau_{st} * J_{st}}{R_s}$

$2T = \frac{100 * \frac{\pi}{32} (50)^4}{\frac{50}{2}}$

$T = 1227184.6$ N.mm

Based on θ

$\frac{T}{J} = \frac{G\theta}{L} \Rightarrow \theta = \frac{TL}{GJ}$

$\theta_T = \theta_{al} + \theta_s$

$= \frac{T_{al} L_{al}}{G_{al} J_{al}} + \frac{T_{st} L_{st}}{G_{st} J_{st}}$

$12 * \frac{\pi}{180} = \frac{3T * 2000}{28*10^3 * \frac{\pi}{32}(75)^4} + \frac{2*T*1500}{83*10^3*\frac{\pi}{32}(50)^4}$

$\Rightarrow T = 1637647.28$ N.mm

The safe value of Torque (1227184.6 N.m)

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