3

Mathematical Modeling of Mechanical Systems and Electrical Systems

3–1 INTRODUCTION

This chapter presents mathematical modeling of mechanical systems and electrical systems. In Chapter 2 we obtained mathematical models of a simple electrical circuit and a simple mechanical system. In this chapter we consider mathematical modeling of a variety of mechanical systems and electrical systems that may appear in control systems.

The fundamental law govering mechanical systems is Newton's second law. In Section 3–2 we apply this law to various mechanical systems and derive transfer-function models and state-space models.

The basic laws governing electrical circuits are Kirchhoff's laws. In Section 3–3 we obtain transfer-function models and state-space models of various electrical circuits and operational amplifier systems that may appear in many control systems.

3–2 MATHEMATICAL MODELING OF MECHANICAL SYSTEMS

This section first discusses simple spring systems and simple damper systems. Then we derive transfer-function models and state-space models of various mechanical systems.

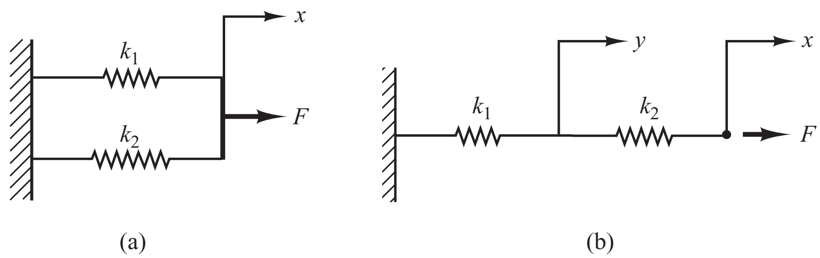


Figure 3–1
(a) System consisting of two springs in parallel;
(b) system consisting of two springs in series.

EXAMPLE 3–1 Let us obtain the equivalent spring constants for the systems shown in Figures 3–1(a) and (b), respectively.

For the springs in parallel [Figure 3–1(a)] the equivalent spring constant k_{eq} is obtained from

$$k_1 x + k_2 x = F = k_{eq} x$$

or

$$k_{eq} = k_1 + k_2$$

For the springs in series [Figure–3–1(b)], the force in each spring is the same. Thus

$$k_1 y = F, \qquad k_2(x - y) = F$$

Elimination of y from these two equations results in

$$k_2\left(x - \frac{F}{k_1}\right) = F$$

or

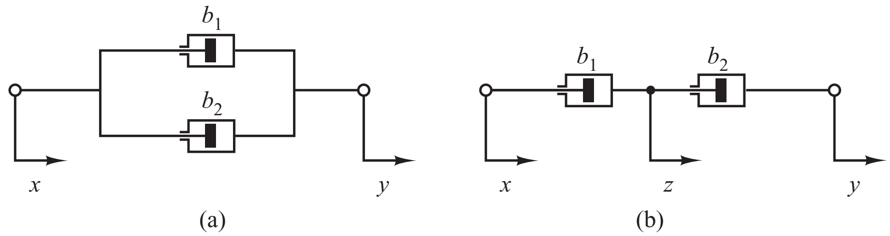
$$k_2 x = F + \frac{k_2}{k_1} F = \frac{k_1 + k_2}{k_1} F$$

The equivalent spring constant k_{eq} for this case is then found as

$$k_{eq} = \frac{F}{x} = \frac{k_1 k_2}{k_1 + k_2} = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

EXAMPLE 3–2 Let us obtain the equivalent viscous-friction coefficient b_{eq} for each of the damper systems shown in Figures 3–2(a) and (b). An oil-filled damper is often called a dashpot. A dashpot is a device that provides viscous friction, or damping. It consists of a piston and oil-filled cylinder. Any relative motion between the piston rod and the cylinder is resisted by the oil because the oil must flow around the piston (or through orifices provided in the piston) from one side of the piston to the other. The dashpot essentially absorbs energy. This absorbed energy is dissipated as heat, and the dashpot does not store any kinetic or potential energy.

Figure 3–2
(a) Two dampers connected in parallel; (b) two dampers connected in series.



(a) The force f due to the dampers is

$$f = b_1(\dot{y} - \dot{x}) + b_2(\dot{y} - \dot{x}) = (b_1 + b_2)(\dot{y} - \dot{x})$$

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

Hence

$$b_{eq} = b_1 + b_2$$

(b) The force f due to the dampers is

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \tag{3–1}$$

where z is the displacement of a point between damper b_1 and damper b_2. (Note that the same force is transmitted through the shaft.) From Equation (3–1), we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x}$$

or

$$\dot{z} = \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x}) \tag{3–2}$$

In terms of the equivalent viscous-friction coefficient b_{eq}, force f is given by

$$f = b_{eq}(\dot{y} - \dot{x})$$

By substituting Equation (3–2) into Equation (3–1), we have

$$\begin{aligned} f = b_2(\dot{y} - \dot{z}) &= b_2\left[\dot{y} - \frac{1}{b_1 + b_2}(b_2\dot{y} + b_1\dot{x})\right] \\ &= \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x}) \end{aligned}$$

Thus,

$$f = b_{eq}(\dot{y} - \dot{x}) = \frac{b_1 b_2}{b_1 + b_2}(\dot{y} - \dot{x})$$

Hence,

$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = \frac{1}{\dfrac{1}{b_1} + \dfrac{1}{b_2}}$$

EXAMPLE 3–3 Consider the spring-mass-dashpot system mounted on a massless cart as shown in Figure 3–3. Let us obtain mathematical models of this system by assuming that the cart is standing still for $t < 0$ and the spring-mass-dashpot system on the cart is also standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\dot{u} =$ constant. The displacement $y(t)$ of the mass is the output. (The displacement is relative to the ground.) In this system, m denotes the mass, b denotes the viscous-friction coefficient, and k denotes the spring constant. We assume that the friction force of the dashpot is proportional to $\dot{y} - \dot{u}$ and that the spring is a linear spring; that is, the spring force is proportional to $y - u$.

For translational systems, Newton's second law states that

$$ma = \sum F$$

where m is a mass, a is the acceleration of the mass, and $\sum F$ is the sum of the forces acting on the mass in the direction of the acceleration a. Applying Newton's second law to the present system and noting that the cart is massless, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

This equation represents a mathematical model of the system considered. Taking the Laplace transform of this last equation, assuming zero initial condition, gives

$$\left(ms^2 + bs + k\right)Y(s) = (bs + k)U(s)$$

Taking the ratio of $Y(s)$ to $U(s)$, we find the transfer function of the system to be

$$\text{Transfer function} = G(s) = \frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Such a transfer-function representation of a mathematical model is used very frequently in control engineering.

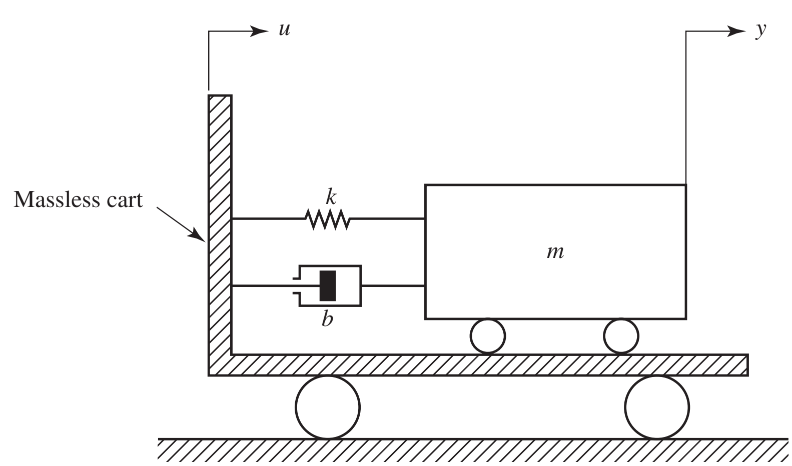


Figure 3–3 Spring-mass-dashpot system mounted on a cart.

Next we shall obtain a state-space model of this system. We shall first compare the differential equation for this system

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

and identify a_1, a_2, b_0, b_1, and b_2 as follows:

$$a_1 = \frac{b}{m}, \qquad a_2 = \frac{k}{m}, \qquad b_0 = 0, \qquad b_1 = \frac{b}{m}, \qquad b_2 = \frac{k}{m}$$

Referring to Equation (3–35), we have

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

Then, referring to Equation (2–34), define

$$x_1 = y - \beta_0 u = y$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

From Equation (2–36) we have

$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u$$

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u$$

and the output equation becomes

$$y = x_1$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ \frac{k}{m} - \left(\frac{b}{m}\right)^2 \end{bmatrix} u \tag{3–3}$$

and

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3–4}$$

Equations (3–3) and (3–4) give a state-space representation of the system. (Note that this is not the only state-space representation. There are infinitely many state-space representations for the system.)

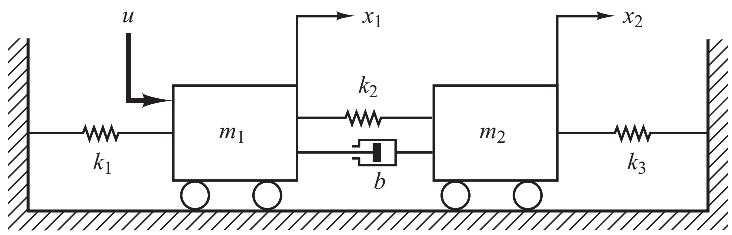


Figure 3–4
Mechanical system.

EXAMPLE 3–4 Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure 3–4.

The equations of motion for the system shown in Figure 3–4 are

$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) + u$$

$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Simplifying, we obtain

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = b\dot{x}_2 + k_2x_2 + u$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2x_1$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$\left[m_1s^2 + bs + (k_1 + k_2)\right]X_1(s) = (bs + k_2)X_2(s) + U(s) \tag{3–5}$$

$$\left[m_2s^2 + bs + (k_2 + k_3)\right]X_2(s) = (bs + k_2)X_1(s) \tag{3–6}$$

Solving Equation (3–6) for $X_2(s)$ and substituting it into Equation (3–5) and simplifying, we get

$$\left[(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2\right]X_1(s)$$
$$= (m_2s^2 + bs + k_2 + k_3)U(s)$$

from which we obtain

$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2 + bs + k_2 + k_3}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \tag{3–7}$$

From Equations (3–6) and (3–7) we have

$$\frac{X_2(s)}{U(s)} = \frac{bs + k_2}{(m_1s^2 + bs + k_1 + k_2)(m_2s^2 + bs + k_2 + k_3) - (bs + k_2)^2} \tag{3–8}$$

Equations (3–7) and (3–8) are the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$, respectively.

EXAMPLE 3–5 An inverted pendulum mounted on a motor-driven cart is shown in Figure 3–5(a). This is a model of the attitude control of a space booster on takeoff. (The objective of the attitude control problem is to keep the space booster in a vertical position.) The inverted pendulum is unstable in that it may fall over any time in any direction unless a suitable control force is applied. Here we consider

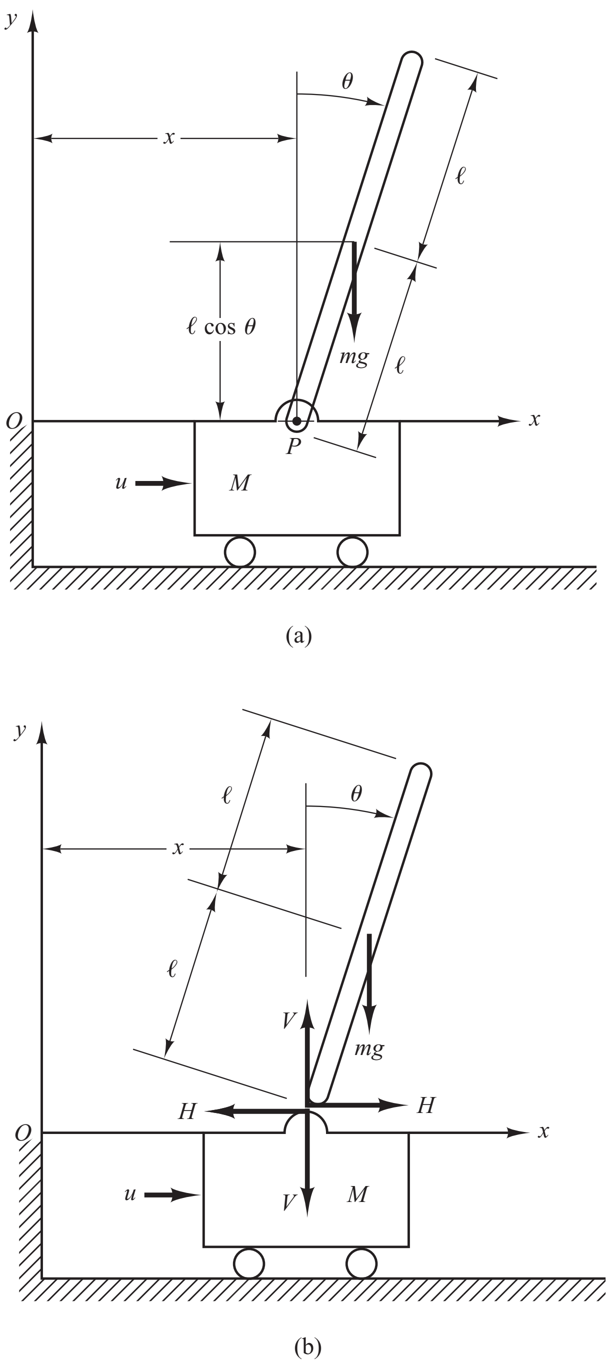


Figure 3–5
(a) Inverted pendulum system; (b) free-body diagram.

only a two-dimensional problem in which the pendulum moves only in the plane of the page. The control force u is applied to the cart. Assume that the center of gravity of the pendulum rod is at its geometric center. Obtain a mathematical model for the system.

Define the angle of the rod from the vertical line as θ. Define also the (x, y) coordinates of the center of gravity of the pendulum rod as (x_G, y_G). Then

$$x_G = x + l \sin\theta$$

$$y_G = l \cos\theta$$

To derive the equations of motion for the system, consider the free-body diagram shown in Figure 3–5(b). The rotational motion of the pendulum rod about its center of gravity can be described by

$$I\ddot{\theta} = Vl\sin\theta - Hl\cos\theta \tag{3–9}$$

where I is the moment of inertia of the rod about its center of gravity.

The horizontal motion of center of gravity of pendulum rod is given by

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \tag{3–10}$$

The vertical motion of center of gravity of pendulum rod is

$$m\frac{d^2}{dt^2}(l\cos\theta) = V - mg \tag{3–11}$$

The horizontal motion of cart is described by

$$M\frac{d^2x}{dt^2} = u - H \tag{3–12}$$

Since we must keep the inverted pendulum vertical, we can assume that $\theta(t)$ and $\dot{\theta}(t)$ are small quantities such that $\sin\theta \doteqdot \theta, \cos\theta = 1$, and $\theta\dot{\theta}^2 = 0$. Then, Equations (3–9) through (3–11) can be linearized. The linearized equations are

$$I\ddot{\theta} = Vl\theta - Hl \tag{3–13}$$

$$m(\ddot{x} + l\ddot{\theta}) = H \tag{3–14}$$

$$0 = V - mg \tag{3–15}$$

From Equations (3–12) and (3–14), we obtain

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{3–16}$$

From Equations (3–13), (3–14), and (3–15), we have

$$\begin{aligned} I\ddot{\theta} &= mgl\theta - Hl \\ &= mgl\theta - l(m\ddot{x} + ml\ddot{\theta}) \end{aligned}$$

or

$$(I + ml^2)\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{3–17}$$

Equations (3–16) and (3–17) describe the motion of the inverted-pendulum-on-the-cart system. They constitute a mathematical model of the system.

EXAMPLE 3–6 Consider the inverted-pendulum system shown in Figure 3–6. Since in this system the mass is concentrated at the top of the rod, the center of gravity is the center of the pendulum ball. For this case, the moment of inertia of the pendulum about its center of gravity is small, and we assume $I = 0$ in Equation (3–17). Then the mathematical model for this system becomes as follows:

$$(M + m)\ddot{x} + ml\ddot{\theta} = u \tag{3–18}$$

$$ml^2\ddot{\theta} + ml\ddot{x} = mgl\theta \tag{3–19}$$

Equations (3–18) and (3–19) can be modified to

$$Ml\ddot{\theta} = (M + m)g\theta - u \tag{3–20}$$

$$M\ddot{x} = u - mg\theta \tag{3–21}$$

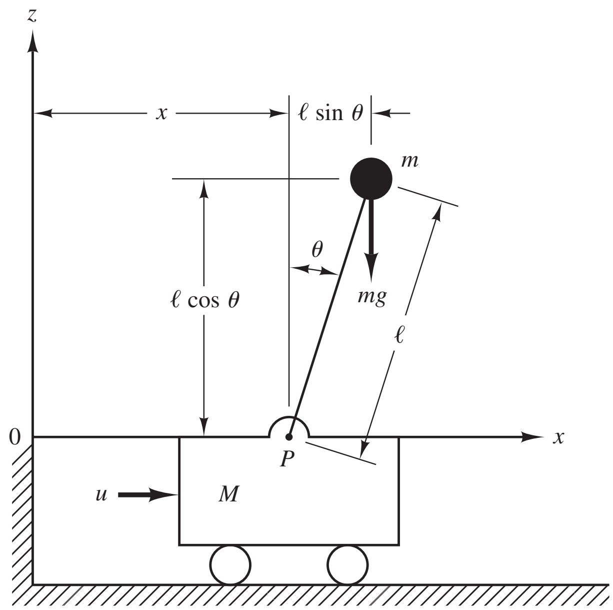


Figure 3–6
Inverted-pendulum system.

Equation (3–20) was obtained by eliminating $\ddot{x}$ from Equations (3–18) and (3–19). Equation (3–21) was obtained by eliminating $\ddot{\theta}$ from Equations (3–18) and (3–19). From Equation (3–20) we obtain the plant transfer function to be

$$\frac{\Theta(s)}{-U(s)} = \frac{1}{Mls^2 - (M + m)g}$$

$$= \frac{1}{Ml\left(s + \sqrt{\frac{M + m}{Ml}g}\right)\left(s - \sqrt{\frac{M + m}{Ml}g}\right)}$$

The inverted-pendulum plant has one pole on the negative real axis $\left[s = -\left(\sqrt{M + m}/\sqrt{Ml}\right)\sqrt{g}\right]$ and another on the positive real axis $\left[s = \left(\sqrt{M + m}/\sqrt{Ml}\right)\sqrt{g}\right]$. Hence, the plant is open-loop unstable.

Define state variables x_1, x_2, x_3, and x_4 by

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = x$$
$$x_4 = \dot{x}$$

Note that angle θ indicates the rotation of the pendulum rod about point P, and x is the location of the cart. If we consider θ and x as the outputs of the system, then

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}$$

(Notice that both θ and x are easily measurable quantities.) Then, from the definition of the state variables and Equations (3–20) and (3–21), we obtain

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{M + m}{Ml} g x_1 - \frac{1}{Ml} u$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{m}{M} g x_1 + \frac{1}{M} u$$

In terms of vector-matrix equations, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{M+m}{Ml}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{m}{M}g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{1}{Ml} \\ 0 \\ \dfrac{1}{M} \end{bmatrix} u \tag{3–22}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{3–23}$$

Equations (3–22) and (3–23) give a state-space representation of the inverted-pendulum system. (Note that state-space representation of the system is not unique. There are infinitely many such representations for this system.)

3–3 MATHEMATICAL MODELING OF ELECTRICAL SYSTEMS

Basic laws governing electrical circuits are Kirchhoff's current law and voltage law. Kirchhoff's current law (node law) states that the algebraic sum of all currents entering and leaving a node is zero. (This law can also be stated as follows: The sum of currents entering a node is equal to the sum of currents leaving the same node.) Kirchhoff's voltage law (loop law) states that at any given instant the algebraic sum of the voltages around any loop in an electrical circuit is zero. (This law can also be stated as follows: The sum of the voltage drops is equal to the sum of the voltage rises around a loop.) A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff's laws to it.

This section first deals with simple electrical circuits and then treats mathematical modeling of operational amplifier systems.

***LRC* Circuit.** Consider the electrical circuit shown in Figure 3–7. The circuit consists of an inductance L (henry), a resistance R (ohm), and a capacitance C (farad). Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \tag{3–24}$$

$$\frac{1}{C}\int i\,dt = e_o \tag{3–25}$$

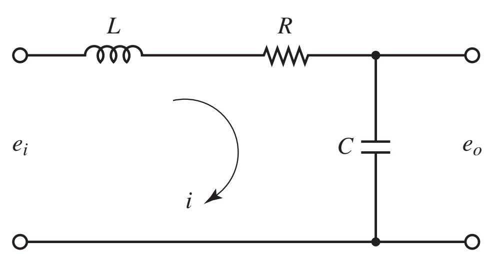


Figure 3–7
Electrical circuit.

Equations (3–24) and (3–25) give a mathematical model of the circuit.

A transfer-function model of the circuit can also be obtained as follows: Taking the Laplace transforms of Equations (3–24) and (3–25), assuming zero initial conditions, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s)$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s)$$

If e_i is assumed to be the input and e_o the output, then the transfer function of this system is found to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1} \tag{3–26}$$

A state-space model of the system shown in Figure 3–7 may be obtained as follows: First, note that the differential equation for the system can be obtained from Equation (3–26) as

$$\ddot{e}_o + \frac{R}{L}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{LC}e_i$$

Then by defining state variables by

$$x_1 = e_o$$
$$x_2 = \dot{e}_o$$

and the input and output variables by

$$u = e_i$$
$$y = e_o = x_1$$

we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} u$$

and

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a mathematical model of the system in state space.

Transfer Functions of Cascaded Elements. Many feedback systems have components that load each other. Consider the system shown in Figure 3–8. Assume that e_i is the input and e_o is the output. The capacitances C_1 and C_2 are not charged initially.

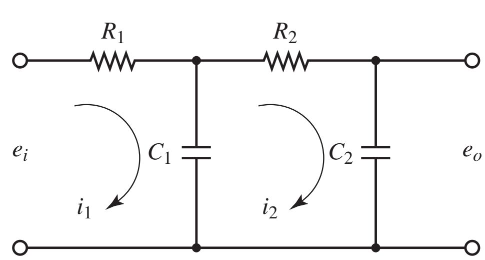


Figure 3–8
Electrical system.