



2.4- Non-Exact First Order Differential Equation

The equation which isn't exact can be made exact by multiplying it by the Integrating Factor (I.F).

ان المعادلة الغير تامة (Non-Exact) نستطيع جعلها تامة بواسطة ضربها بمعامل التكامل، هو معامل يضرب به المعادلة الغير تامة فيحولها الى معادلة تامة.

For Example:

$$2xy^3 dx + 3x^2y^2 dy = 0$$

$$M = 2xy^3, N = 3x^2y^2$$

$$\left. \frac{\partial M}{\partial y} = 6xy^2, \frac{\partial N}{\partial x} = 6xy^2 \right\} \rightarrow \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{Exact}$$

$$\text{if } 2xy^3 dx + 3x^2y^2 dy = 0 \text{] } \div x$$

$$2y^3 dx + 3xy^2 dy = 0$$

$$\left. \frac{\partial M}{\partial y} = 6y^2, \frac{\partial N}{\partial x} = 3y^2 \right\} \rightarrow \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore \text{Non - Exact}$$

لو كانت العملية بالعكس فان معامل التكامل هو (x) عند ضرب المعادلة غير التامة يعمل على تحويلها الى معادلة تامة.

$\therefore x$ is Integrating Factor

$$\text{if } 2xy^3 dx + 3x^2y^2 dy = 0 \text{] } \div y^2$$

$$2xy dx + 3x^2 dy = 0$$

$$\left. \frac{\partial M}{\partial y} = 2x, \frac{\partial N}{\partial x} = 6x \right\} \rightarrow \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore \text{Non - Exact}$$

$\therefore y^2$ is integrating factor

How to Find Integrating Factor:

If the integrating factor is a function of (x) only (r);

$$M(x, y) \cdot dx + N(x, y) \cdot dy = 0 \} * r$$

$$r M(x, y) \cdot dx + r N(x, y) \cdot dy = 0$$

$$\frac{\partial(rM)}{\partial y} = \frac{\partial(rN)}{\partial x}$$

$$r \frac{\partial M}{\partial y} = r \frac{\partial N}{\partial x} + N \frac{dr}{dx} \rightarrow \text{re - arrangement}$$

$$r \frac{\partial M}{\partial y} - r \frac{\partial N}{\partial x} = N \frac{dr}{dx}$$

$$r \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = N \frac{dr}{dx}$$

$$\frac{dr}{r} = \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx \rightarrow \int \frac{dr}{r} = \int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx$$

$$\ln r = \int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx \quad \text{نأخذ } e \text{ للطرفين}$$

$$\therefore r_x = e^{\int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx} \quad \text{integrating factor is a function of (x)}$$

by the same method get to:

$$r_y = e^{\int \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy} \quad \text{integrating factor is a function of (y)}$$

Example (1): Find a general solution for $y \cdot dx + (3 + 3x - y) \cdot dy = 0$?

Solve:

$$M = y, \quad N = (3 + 3x - y)$$

$$\left. \frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = 3 \right\} \rightarrow \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore \text{Non - Exact}$$

\therefore Find integrating factor:

$$\begin{aligned}\therefore r_x &= e^{\int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx} = e^{\int \frac{1}{(3+3x-y)} (1-3) dx} = e^{\int \frac{-2}{(3+3x-y)} \times \frac{3}{3} dx} \\ &= e^{-\frac{2}{3} \ln(3+3x-y)} = e^{\ln(3+3x-y)^{-2/3}} = (3+3x-y)^{-2/3}\end{aligned}$$

يهمل هذا المعامل لأنه دالة من x و y .

$$r_y = e^{\int \frac{1}{M} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy} = e^{\int \frac{1}{y} (3-1) dy} = e^{\int \frac{2}{y} dy} = e^{2 \ln y} = e^{\ln y^2}$$

$$r_y = y^2$$

$\therefore y^2$ is integrating factor

$$y \cdot dx + (3 + 3x - y) \cdot dy = 0 \} * y^2$$

$$y^3 \cdot dx + (3y^2 + 3xy^2 - y^3) \cdot dy = 0$$

$$M = y^3, \quad N = (3y^2 + 3xy^2 - y^3)$$

$$\left. \begin{aligned} \frac{\partial M}{\partial y} &= 3y^2, & \frac{\partial N}{\partial x} &= 3y^2 \end{aligned} \right\} \rightarrow \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{Exact}$$

To find $f(x, y)$:

$$M = \frac{df}{dx} = y^3 \rightarrow \int df = \int y^3 \cdot dx$$

$$\therefore f(x, y) = xy^3 + g(y)$$

$$\frac{df}{dy} = 3xy^2 + \frac{dg}{dy} \dots \dots \dots (1)$$

$$\therefore N = \frac{df}{dy} = 3y^2 + 3xy^2 - y^3 \dots \dots \dots (2)$$

Sub eq. (1) and eq. (2) to get:

$$3xy^2 + \frac{dg}{dy} = 3y^2 + 3xy^2 - y^3 \rightarrow$$

$$\frac{dg}{dy} = 3y^2 - y^3 \rightarrow \int dg = \int (3y^2 - y^3) \cdot dy + c$$

$$g(y) = y^3 - \frac{y^4}{4} + c$$

$$\therefore f(x, y) = xy^3 + y^3 - \frac{y^4}{4} + c$$

Example (2): Show that $\frac{1}{x^2+y^2}$ is an integrating factor for the equation
 $(x^2 + y^2 - x) \cdot dx - y \cdot dy = 0$?

Solve:

$$M = (x^2 + y^2 - x), \quad N = -y$$

$$\left. \begin{array}{l} \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 0 \end{array} \right\} \rightarrow \because \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \therefore \text{Non-Exact}$$

$$(x^2 + y^2 - x) \cdot dx - y \cdot dy = 0 \quad * \quad \frac{1}{x^2 + y^2}$$

$$\left(1 - \frac{x}{x^2 + y^2}\right) \cdot dx - \frac{y}{x^2 + y^2} \cdot dy = 0$$

$$\left. \begin{array}{l} \frac{\partial M}{\partial y} = \left(0 - \frac{0 - x \cdot 2y}{(x^2 + y^2)^2}\right) = \frac{2xy}{(x^2 + y^2)^2} \\ \frac{\partial N}{\partial x} = \left(-\frac{0 - y \cdot 2x}{(x^2 + y^2)^2}\right) = \frac{2xy}{(x^2 + y^2)^2} \end{array} \right\} \rightarrow \because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \text{Exact}$$

$\therefore \frac{1}{x^2 + y^2}$ is integrating factor

Problems:

H.W- Check the given expression is **Exact** or **Non-Exact** differential equation. if the expression is **Non-Exact** equation, Find the Integration Factor (**I.F**) for:

1) $(x - y) \cdot dx - dy = 0$

Answer: Non-Exact, I.F: e^x

2) $(y^2 - 1) \cdot dx + (2xy - \sin y) \cdot dy = 0$

Answer: Exact

3) $y \cdot dx + (4x - y^2) \cdot dy = 0$

Answer: Non-Exact, I.F: y^3

4) $(y^2 - x^2) \cdot dy - 2xy \cdot dx = 0$

Answer: Exact

5) $(2y \sin x - 3) \cdot dx - \cos x \cdot dy = 0$

Answer: Non-Exact, I.F: $\cos x$