Example (3): Find the complete solution of the differential equation:

$$\acute{\acute{y}} + 2\,\acute{y} + y = e^{-x}\ \ln(x)$$

Solve:

$$\acute{\acute{y}} + 2\,\acute{y} + y = 0$$

$$m^2 + 2\,m + 1 = 0$$

$$(m+1)(m+1) = 0$$

$$\therefore m_1 = m_2 = m = -1$$

$$y_c = c_1\,e^{mx} + c_2\,x\,e^{m\,x} \quad \rightarrow$$

$$y_c = c_1\,e^{-x} + c_2\,x\,e^{-x}$$

$$Y = u_1\,y_1 + u_2\,y_2 = u_1\,e^{-x} + u_2\,x\,e^{-x}$$

$$y_1 = e^{-x},\ \ y_2 = x\,e^{-x}\,, R(x) = e^{-x}\ \ln(x)$$

$$u_1 = \int \frac{-y_2}{(y_1\acute{y}_2 - \acute{y}_1 y_2)}\,R(x)\,dx$$

$$u_1 = \int \frac{-x\,e^{-x}}{(e^{-x}(-x\,e^{-x} + e^{-x}) - (-e^{-x})x\,e^{-x})}\,e^{-x}\ \ln(x)\cdot dx$$

$$u_1 = \int \frac{-x\,e^{-2x}}{-x\,e^{-2x} + e^{-2x} + x\,e^{-2x}}\ \ln(x)\cdot dx = \int -x\ \ln(x)\cdot dx$$

$\int -x\ \ln(x)\cdot dx$ use $(u\cdot dv)$ to solve integral

$$\int u\cdot dv = u\cdot v - \int v\cdot du$$

$$let\ \ u = \ln(x) \quad \rightarrow \quad du = \frac{1}{x}\,dx$$

$$dv = -x\cdot\ dx \quad \rightarrow \quad v = -\frac{x^2}{2}$$

$$u_1 = \int -x\ \ln(x)\cdot dx = -\frac{x^2}{2}\ \ln(x) - \int -\frac{x^2}{2}\cdot\frac{1}{x}\,dx = -\frac{x^2}{2}\ \ln(x) + \int \frac{x}{2}\,dx$$

$$u_1 = -\frac{x^2}{2}\ln(x) + \frac{x^2}{4}$$

$$u_2 = \int \frac{y_1}{(y_1\acute{y}_2 - \acute{y}_1 y_2)} R(x)\cdot dx$$

$$u_2 = \int \frac{e^{-x}}{(e^{-x}(-x\,e^{-x} + e^{-x}) - (-e^{-x})x\,e^{-x})} e^{-x}\ \ln(x)\cdot dx$$

$$u_2 = \int \frac{e^{-2x}}{-x\,e^{-2x} + e^{-2x} + x\,e^{-2x}}\ \ln(x)\cdot dx = \int \ln(x)\cdot dx$$

$\int \ln(x)\cdot dx$ use $(u\cdot dv)$ to solve integral

$$\int u\cdot dv = u\cdot v - \int v\cdot du$$

$let\ \ u = \ln(x) \quad \rightarrow \quad du = \frac{1}{x}\,dx$

$dv = 1\cdot\ dx \quad \rightarrow \quad v = x$

$$u_2 = \int \ln(x)\cdot dx = x\ln(x) - \int x\cdot\frac{1}{x}\,dx = x\ln(x) - \int 1\,dx$$

$$u_2 = x\ln(x) - x$$

$$\therefore Y = \left(-\frac{x^2}{2}\ln(x) + \frac{x^2}{4}\right) e^{-x} + (x\ln(x) - x)\,x\,e^{-x}$$

$$Y = x^2 e^{-x}\left(\frac{1}{2}\ln(x) - \frac{3}{4}\right)$$

$\therefore$ The complete solution: $\quad y = c_1\,e^{-x} + c_2\,x\,e^{-x} + x^2 e^{-x}\left(\frac{1}{2}\ln(x) - \frac{3}{4}\right)$

H.W: Find the complete solution of the differential equation:

1) $\acute{\acute{y}} - 2\,\acute{y} + y = x^3\,e^x$

Ans: $y = c_1\,e^x + c_2 x\,e^x + \frac{1}{20}x^5\,e^x$