2.2 Homogeneous First Order Differential Equation

تكون المعادلة التفاضلية متجانسة إذا كانت الدالة $M(x, y)$ متجانسة والدالة $N(x, y)$ متجانسة أيضا وبنفس الدرجة. ويتم التحقق من ذلك من خلال تعويض عن كل x ب (λx) وعن كل y ب (λy) فتحقق ما يلي:

$\boldsymbol{M(x, y)dx = N(x, y)dy}$ *is homogenous equation if*:

$M(\lambda x, \lambda y) = \lambda^n \cdot M(x, y)$ $\quad \therefore$ *This part is homogenous*

$N(\lambda x, \lambda y) = \lambda^n \cdot N(x, y)$ $\quad \therefore$ *This part is homogenous*

$\therefore$ The total equation is homogenous.

يجب ان تكون درجة λ^n متشابهة في الدالة $M(x, y)$ والدالة $N(x, y)$.

To solve this equation, always assume:

$y = u \cdot x$

$dy = u \cdot dx + x \cdot du$

عند اجراء هذا التعويض فأن المعادلة التفاضلية دائما تتحول الى دالة قابلة للفصل وبذلك يسهل حلها. وبعد الحصول على الحل نعوض بدل كل u :

$$u = \frac{y}{x}$$

Example (1): Prove that the function is homogenous:

$$\frac{dy}{dx} = \frac{x - y}{x + y}$$

Solve:

$(x + y)\, dy = (x - y)dx$

$M(x, y) = (x - y)$

$M(\lambda x, \lambda y) = (\lambda x - \lambda y)$

$= \lambda(x - y) = \lambda\, M(x, y) \quad \therefore hom.$

$N(x,y) = (x+y)$

$N(\lambda x,\lambda y) = (\lambda x+\lambda y) = \lambda(x+y) = \lambda\, N(x,y) \quad \therefore hom.$

$\therefore$ The total equation is homogenous.

Example (2): Find the general solution for $(x^2+3y^2)dx = 2xy\, dy$:

Solve:

$M(x,y) = (x^2+3y^2)$

$M(\lambda x,\lambda y) = (\lambda^2 x^2 + 3\lambda^2 y^2)$

$= \lambda^2(x^2+3y^2) = \lambda^2\, M(x,y) \quad \therefore hom.$

$N(x,y) = 2xy$

$N(\lambda x,\lambda y) = 2\lambda x\cdot\lambda y = \lambda^2\cdot 2xy = \lambda^2\, N(x,y) \quad \therefore hom.$

$\therefore$ The total equation is homogenous.

To solve this equation, assume:

$y = u\cdot x \quad and \quad dy = u\cdot dx + x\cdot du$

$(x^2+3u^2x^2)dx = 2ux^2\,(u\cdot dx + x\cdot du)$

$x^2\, dx + 3u^2x^2\, dx = 2u^2x^2\, dx + 2ux^3\, du$

$x^2\, dx + u^2x^2\, dx = 2ux^3\, du$

$(1+u^2)x^2\, dx = 2ux^3\, du \quad \rightarrow \quad \text{re} - \text{arrangement:}$

$$\left.\frac{dx}{x} = \frac{2u}{(1+u^2)}\, du\right\}\ by\ integral$$

$ln\, x = ln(1+u^2) + c \quad \rightarrow \quad ln\, x - ln(1+u^2) = c$

$\ln \frac{x}{(1+u^2)} = c$ نأخذ e للطرفين

$\frac{x}{(1+u^2)} = e^c = k$

$but\ u = \frac{y}{x}\ lead\ to:$

$\frac{x}{1+\frac{y^2}{x^2}} = k \quad \rightarrow \quad \frac{x^3}{x^2+y^2} = k$

Example (3): Find the general solution for $y^2dx + x^2dy = 2xy\,dy$?

Solve:

$y^2dx = 2xy\,dy - x^2dy$

$y^2dx = (2xy - x^2)\,dy$

$M(x,y) = y^2$

$M(\lambda x, \lambda y) = \lambda^2 y^2$

$= \lambda^2\,M(x,y) \quad \therefore hom.$

$N(x,y) = (2xy - x^2)$

$N(\lambda x, \lambda y) = (2\lambda x\,\lambda y - \lambda^2 x^2) = \lambda^2 \cdot (2xy - x^2)$

$= \lambda^2\,N(x,y) \quad \therefore hom.$

$\therefore$ The total equation is homogenous.

To solve this equation, assume:

$y = u \cdot x \quad and \quad dy = u \cdot dx + x \cdot du$

$u^2x^2dx = (2ux^2 - x^2)\,(u \cdot dx + x \cdot du)$

$u^2x^2\,dx = 2u^2x^2\,dx + 2ux^3\,du - ux^2\,dx - x^3\,du$

$ux^2\,dx - u^2x^2\,dx = 2ux^3\,du - x^3\,du$

$(u-u^2)x^2\,dx = x^3(2u-1)\,du \quad \rightarrow \quad \text{re}-\text{arrangement:}$

$$\frac{1}{x}dx = \frac{(2u-1)}{(u-u^2)}\,du \;\times \frac{-1}{-1}\Big\}\; by\ integral$$

$$ln\,x = -\,ln(u-u^2)+c \quad \rightarrow \quad ln\,x + ln(u-u^2) = c$$

$\ln\big(x(u-u^2)\big) = c$ نأخذ e للطرفين

$xu - xu^2 = e^c = k$

$but\ u = \dfrac{y}{x}\ \ lead\ to:$

$$x\frac{y}{x} - x\frac{y^2}{x^2} = k \quad \rightarrow \quad y - \frac{y^2}{x} = k\Big\}\times x$$

$$xy - y^2 - kx = 0$$

Problems:

1. Find a particular solution for $(3y^3 - x^3)dx = 3xy^2\,dy$, if $x = 1, y = 2.$

Answer: $y^3 = x^3\,(8 - \ln x)$

2. Solve $2x\,(x+y)dx + (x^2+y^2)\,dy = 0$

Answer: $2x^3 + 3x^2y + y^3 = k$