4- Linear Ordinary Differential Equations with Constant Coefficients

The general form:

$$\acute{\acute{y}} + P(x)\acute{y} + Q(x)y = R(x)$$

if $R(x) = 0$ $\quad\therefore$ Homogenous Equations معادلة متجانسة

if $R(x) \neq 0$ $\quad\therefore$ Non-Homogenous Equations معادلة غير متجانسة

where:

$P(x)$: The function adjacent to $(\acute{y})$ when the coefficient of $\acute{\acute{y}}$ is equal to 1.

$P(x)$: هو الدالة المجاورة لل $\acute{y}$ عندما معامل $\acute{\acute{y}}$ يساوي 1.

$Q(x)$: The function adjacent to (y) when the coefficient of $\acute{\acute{y}}$ is equal to 1.

$Q(x)$: هو الدالة المجاورة لل y عندما معامل $\acute{\acute{y}}$ يساوي 1.

$R(x)$: The right side function is free from (y) and its derivatives when the coefficient of $\acute{\acute{y}}$ is equal to 1.

$R(x)$: هو الدالة في الجهة اليمنى و الخالية من المتغير y ومشتقاته عندما معامل $\acute{\acute{y}}$ يساوي 1.

4.1- Homogeneous Second Order Linear Differential Equations

Theorem: if y_1 and y_2 are two solutions to the homogenous equation, then $y_c = c_1\, y_1 + c_2\, y_2$ is general solution for Homogenous Equations, where c_1 and c_2 are constant.

اذا كان y_1 و y_2 حلين للمعادلة التفاضلية المتجانسة فان $y_c = c_1\, y_1 + c_2\, y_2$ هو الحل العام لهذه المعادلة.
معادلة الحل العام هو:

$$y_c = c_1\, y_1 + c_2\, y_2$$

في حالة توفر احد الحلول (y_1) نستطيع ان نحصل على الحل الثاني (y_2) من خلال هذه المعادلة:

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{{y_1}^2}\, dx$$

وبالتالي يكون الحل العام لمثل هذه الحالة كما يلي:

$$\therefore y_c = c_1\, y_1 + c_2\, y_1 \int \frac{e^{-\int P(x)dx}}{{y_1}^2}\, dx$$

Example (1): Find the general solution if $y_1 = x^2$ is a solution of the equation:

$$x^2\, \acute{\acute{y}} + x\, \acute{y} - 4y = 0$$

Solve:

$$x^2\, \acute{\acute{y}} + x\, \acute{y} - 4y = 0] \div x^2$$

$$\acute{\acute{y}} + \frac{1}{x}\, \acute{y} - \frac{4}{x^2} y = 0$$

The general equation: $\acute{\acute{y}} + P(x)\acute{y} + Q(x)y = 0$

$$\therefore\ P(x) = \frac{1}{x}, \quad y_1 = x^2$$

$$\because y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{{y_1}^2}\, dx$$

$$y_2 = x^2 \int \frac{e^{-\int \frac{1}{x} dx}}{x^4}\, dx = x^2 \int \frac{e^{-\ln x}}{x^4}\, dx = x^2 \int \frac{e^{\ln x^{-1}}}{x^4}\, dx$$

$$y_2 = x^2 \int x^{-1} \cdot x^{-4}\, dx = x^2 \int x^{-5}\, dx = x^2 \cdot \frac{x^{-4}}{-4} = x^2 \cdot \frac{-1}{4\, x^4}$$

$$\therefore y_2 = \frac{-1}{4\, x^2}$$

$$y_c = c_1\, y_1 + c_2\, y_2$$

$$y_c = c_1\, x^2 + c_2\, \frac{-1}{4\, x^2}$$

$$\therefore\ y_c = c_1\, x^2 - \frac{c_2}{4\, x^2}$$

Example (2): Find the general solution for the equation $\acute{\acute{y}} - 2\,\acute{y} - 3y = 0$ if $y_1 = e^{3x}$?

Solve:

$\acute{\acute{y}} - 2\,\acute{y} - 3y = 0$

The general equation: $\acute{\acute{y}} + P(x)\acute{y} + Q(x)y = 0$

$\therefore\ P(x) = -2, \quad y_1 = e^{3x}$

$$\because y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{{y_1}^2}\,dx$$

$$y_2 = e^{3x} \int \frac{e^{-\int -2\,dx}}{(e^{3x})^2}\,dx = e^{3x} \int \frac{e^{2x}}{e^{6x}}\,dx = e^{3x} \int e^{2x} \cdot e^{-6x}\,dx$$

$$y_2 = e^{3x} \int e^{-4x}\,dx = e^{3x} \cdot \frac{e^{-4x}}{-4}$$

$$\therefore y_2 = \frac{-1}{4} e^{-x}$$

$$y_c = c_1\,y_1 + c_2\,y_2$$

$$y_c = c_1\,e^{3x} + c_2\,\frac{-1}{4} e^{-x}$$

$$\therefore\ y_c = c_1\,e^{3x} - \frac{c_2}{4} e^{-x}$$

H.W: Using the given solution find a general solution of each of the following equation:

1) $\acute{\acute{y}} + y = 0, \quad y_1 = \sin x$

Ans: $y_c = c_1\ \sin x - c_2\ \cos x$

2) $x^2\,\acute{\acute{y}} + (\,x^2 - 2x)\,\acute{y} + (x+2)y = 0, \quad y_1 = x$

Ans: $y_c = c_1\,x - c_2\,x\,e^{-x}$