4.2.2 Method of Variation of Parameters

تستخدم هذه الطريقة عندما تكون الدالة في الطرف الأيمن من المعادلة التفاضلية تمتلك عدد محدد او غير محدد من المشتقات المستقلة خطيا.

$\acute{\acute{y}} + P(x)\acute{y} + Q(x)y = R(x)$

$y = y_c + Y$ حل المعادلة

$y_c = c_1\, y_1 + c_2\, y_2$

نفرض ال(Y) دالة تشبه حل المعادلة المتجانسة مع تغيير الثوابت.

$let\ \boldsymbol{Y = u_1\, y_1 + u_2\, y_2}$

$\acute{Y} = u_1\, \acute{y}_1 + \acute{u}_1\, y_1 + u_2\, \acute{y}_2 + \acute{u}_2\, y_2$

لتبسيط الحل نفرض:

$\acute{u}_1\, y_1 + \acute{u}_2\, y_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

$\therefore\ \acute{Y} = u_1\, \acute{y}_1 + u_2\, \acute{y}_2$

$\acute{\acute{Y}} = u_1\, \acute{\acute{y}}_1 + \acute{u}_1\, \acute{y}_1 + u_2\, \acute{\acute{y}}_2 + \acute{u}_2\, \acute{y}_2$

Sub $Y, \acute{Y}, \acute{\acute{Y}}$ in Non − homogenous equations:

$\left(u_1\, \acute{\acute{y}}_1 + \acute{u}_1\, \acute{y}_1 + u_2\, \acute{\acute{y}}_2 + \acute{u}_2\, \acute{y}_2\right) + P(x)(u_1\, \acute{y}_1 + u_2\, \acute{y}_2) + Q(x)(u_1\, y_1 + u_2\, y_2) = R(x)$

$u_1\left(\acute{\acute{y}}_1 + P(x)\acute{y}_1 + Q(x)y_1\right) + u_2\left(\acute{\acute{y}}_2 + P(x)\acute{y}_2 + Q(x)y_2\right) + \acute{u}_1\, \acute{y}_1 + \acute{u}_2\, \acute{y}_2 = R(x)$

$but \quad \acute{\acute{y}}_1 + P(x)\acute{y}_1 + Q(x)y_1 = 0 \quad and \quad \acute{\acute{y}}_2 + P(x)\acute{y}_2 + Q(x)y_2 = 0$

$\acute{u}_1\, \acute{y}_1 + \acute{u}_2\, \acute{y}_2 = R(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots. (2)$

to find $\acute{u}_1$ and $\acute{u}_2$:

$from\ equ. (1)\ \acute{u}_1 = \dfrac{-\acute{u}_2\, y_2}{y_1} \quad sub\ in\ equ. (2):$

$\left.\dfrac{-\acute{u}_2\, y_2}{y_1}\, \acute{y}_1 + \acute{u}_2\, \acute{y}_2 = R(x) \quad \right] \times y_1$

$\acute{u}_2\, (y_1\acute{y}_2 - \acute{y}_1 y_2\,) = R(x)y_1$

بعد ترتيب المعادلة و التعويض نحصل على:

$$\acute{u}_1 = \frac{-y_2}{(y_1\acute{y}_2 - \acute{y}_1 y_2)} R(x) \quad \rightarrow \quad \boldsymbol{u_1} = \int \frac{-\boldsymbol{y_2}}{(\boldsymbol{y_1\acute{y}_2 - \acute{y}_1 y_2})} \boldsymbol{R(x)\ dx}$$

$$\acute{u}_2 = \frac{y_1}{(y_1\acute{y}_2 - \acute{y}_1 y_2)} R(x) \quad \rightarrow \quad \therefore \boldsymbol{u_2} = \int \frac{\boldsymbol{y_1}}{(\boldsymbol{y_1\acute{y}_2 - \acute{y}_1 y_2})} \boldsymbol{R(x)\ dx}$$

$$\boldsymbol{Y = u_1\, y_1 + u_2\, y_2}$$

Example (1): Find the complete solution of the differential equation:

$$\boldsymbol{\acute{\acute{y}} + y = \sec x}$$

Solve:

$$\acute{\acute{y}} + y = 0$$

$$m^2 + 1 = 0 \quad \rightarrow \quad m^2 = -1$$

$$\therefore m_{1,2} = \pm i = P \pm qi$$

$$\therefore y_c = e^{Px}(A\cos qx + B\sin qx) = e^{0x}(A\cos x + B\sin x)$$

$$\therefore y_c = A\cos x + B\sin x$$

$$Y = u_1\, y_1 + u_2\, y_2 = u_1\ \cos x + u_2\ \sin x$$

$$y_1 = \cos x, \quad y_2 = \sin x\ , R(x) = \sec x$$

$$u_1 = \int \frac{-y_2}{(y_1\acute{y}_2 - \acute{y}_1 y_2)} R(x)\, dx$$

$$u_1 = \int \frac{-\sin x}{(\cos x\cos x - (-\sin x)\sin x)} \sec x \cdot dx = \int \frac{-\sin x}{1} \times \frac{1}{\cos x} \cdot dx$$

$$u_1 = \int \frac{-\sin x}{\cos x} \cdot dx \quad \rightarrow \quad u_1 = \ln(\cos x)$$

$$u_2 = \int \frac{y_1}{(y_1\acute{y}_2 - \acute{y}_1 y_2)} R(x) \cdot dx$$

$$u_2 = \int \frac{\cos x}{(\cos x\cos x - (-\sin x)\sin x)} \sec x \cdot dx = \int \frac{\cos x}{1} \times \frac{1}{\cos x} \cdot dx$$

$$u_2 = \int 1 \cdot dx \quad \rightarrow \quad u_2 = x$$

$$\therefore Y = \ln(\cos x)\ \cos x + x\ \sin x$$

$\therefore$ The complete solution: $\quad y = A\cos x + B\sin x + \ln(\cos x)\ \cos x + x\ \sin x$

Example (2): Find the complete solution of the differential equation:

$$\acute{\acute{y}} - 3\acute{y} + 2y = e^{3x}$$

Solve:

$\acute{\acute{y}} - 3\acute{y} + 2y = 0$

$m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0 \qquad \Longrightarrow \qquad \therefore m_1 = 2, \quad m_2 = 1$

$y_c = c_1\, e^{m_1 x} + c_2\, e^{m_2 x}$

$\therefore y_c = c_1\, e^{2x} + c_2\, e^{x}$

$Y = u_1\, y_1 + u_2\, y_2 = u_1\, e^{2x} + u_2\, e^{x}$

$y_1 = e^{2x}, \;\; y_2 = e^{x}\,, R(x) = e^{3x}$

$$u_1 = \int \frac{-y_2}{(y_1\acute{y}_2 - \acute{y}_1 y_2)}\, R(x)\, dx$$

$$u_1 = \int \frac{-e^{x}}{(e^{2x}e^{x} - 2e^{2x}e^{x})}\, e^{3x}\cdot dx = \int \frac{-1}{-e^{2x}}\, e^{3x}\cdot dx = \int e^{x}\cdot dx$$

$u_1 = e^{x}$

$$u_2 = \int \frac{y_1}{(y_1\acute{y}_2 - \acute{y}_1 y_2)}\, R(x)\cdot dx$$

$$u_2 = \int \frac{e^{2x}}{(e^{2x}e^{x} - 2e^{2x}e^{x})}\, e^{3x}\cdot dx = \int \frac{1}{-e^{x}} e^{3x}\cdot dx = \int -e^{2x}\cdot dx$$

$$u_2 = -\frac{1}{2}e^{2x}$$

$$\therefore Y = e^{x}\, e^{2x} - \frac{1}{2}e^{2x}\, e^{x} = \frac{1}{2}e^{3x}$$

$\therefore$ The complete solution: $\quad y = c_1\, e^{2x} + c_2\, e^{x} + \frac{1}{2}e^{3x}$

H.W: Find the complete solution of the differential equation:

1) $\acute{\acute{y}} + \; y = \sin^2 x$

Ans: $y = A\cos x + B\sin x + \left(\cos x - \frac{1}{3}\cos^3 x\right)\; \cos x + \frac{1}{3}\sin^4 x$

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