Example (3): Find the complete solution of the differential equation:

$$(\boldsymbol{D^4 + 3D^3 + 3D^2 + D})\boldsymbol{y} = \boldsymbol{2x + 8}$$

Solve:

$$m^4 + 3m^3 + 3m^2 + m = 0$$

$$m\,(m^3 + 3m^2 + 3m + 1) = 0$$

$$m_1 = 0$$

$$m^3 + 3m^2 + 3m + 1 = 0$$

by try and error get to:

$$let\ m = -1 \quad \rightarrow \quad -1 + 3 - 3 + 1 = 0 \qquad \therefore ok$$

$$(m + 1)(m^2 + 2m + 1) = 0$$

$$(m + 1)(m + 1)(m + 1) = 0$$

$$m_{2,3,4} = -1$$

$$y_c = c_1 e^0 + c_2\, e^{-x} + c_3 x e^{-x} + c_4 x^2 e^{-x}$$

$$Let\ \ Y = Ax^2 + Bx$$

$$\acute{Y} = 2Ax + B$$

$$\acute{\acute{Y}} = 2A, \quad \acute{\acute{\acute{Y}}} = 0, \quad \acute{\acute{\acute{\acute{Y}}}} = 0$$

$$6A + 2Ax + B = 2x + 8 \qquad \rightarrow$$

$$2A = 2 \qquad \rightarrow \qquad A = 1$$

$$6A + B = 8 \qquad \rightarrow \qquad B = 2$$

$$\begin{array}{r|l} m+1 & m^2 + 2m + 1 \\ \hline & m^3 + 3m^2 + 3m + 1 \\ & m^3 + m^2 \\ \hline & 2m^2 + 3m + 1 \\ & 2m^2 + 2m \\ \hline & m + 1 \\ & m + 1 \\ \hline & 0 \end{array}$$

$Y = x^2 + 2x$

$y = y_c + Y$

∴ The complete solution: $y = c_1 e^0 + c_2\, e^{-x} + c_3 x e^{-x} + c_4 x^2 e^{-x} + x^2 + 2x$

Example (4): Find the general solution of the differential equation:

$$(\boldsymbol{D^4 - 16})\boldsymbol{y = 0}$$

Solve:

$m^4 - 16 = 0$

$(m^2 - 4)(m^2 + 4) = 0 \quad \rightarrow$

$m^2 - 4 = 0 \quad \rightarrow \quad m^2 = 4 \quad \rightarrow \quad m_{1,2} = \pm 2 \quad m_1 \neq m_2$

$m^2 + 4 = 0 \quad \rightarrow \quad m^2 = -4 \quad \rightarrow \quad m_{3,4} = \pm 2i \quad m_3, m_4$ Imaginary

$y_c = c_1 e^{m_1 x} + c_2\, e^{m_2 x} + e^{Px}(c_3 \cos qx + c_4 \sin qx)$

$y_c = c_1 e^{2x} + c_2\, e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$

Example (5): Find the general solution of the differential equation:

$$(\boldsymbol{D^8 - 2D^4 + 1})\boldsymbol{y = 0}$$

Solve:

$m^8 - 2m^4 + 1 = 0$

$(m^4 - 1)(m^4 - 1) = 0 \quad \rightarrow$

$(m^2 - 1)(m^2 + 1)(m^2 - 1)(m^2 + 1) = 0$

$(m - 1)(m + 1)(m^2 + 1)(m - 1)(m + 1)(m^2 + 1) = 0$

$m - 1 = 0 \quad \rightarrow \quad m_1 = 1$

$m + 1 = 0 \quad \rightarrow \quad m_2 = -1$

$m^2 + 1 = 0 \quad \rightarrow \quad m^2 = -1 \quad \rightarrow \quad m_{3,4} = \pm i$

$m - 1 = 0 \quad \rightarrow \quad m_5 = 1$

$m + 1 = 0 \quad \rightarrow \quad m_6 = -1$

$m^2 + 1 = 0 \quad \rightarrow \quad m^2 = -1 \quad \rightarrow \quad m_{7,8} = \pm i$

$m_1 = m_5 = 1$

$m_2 = m_6 = -1$

$m_{3,4} = m_{7,8} = \pm i$

$y_c = c_1 e^x + c_2\, e^{-x} + c_3 \cos x + c_4 \sin x + c_5 x e^x + c_6\, x e^{-x} + c_7 x \cos x + c_8 x \sin x$