Linear Differential Equations 1^{st} Order

It should have the general form

$$y' + P(x)\, y = q(x)$$

To solve

1. اعادة ترتيب للحصول على الصيغة

$$\frac{dy}{dx} + P(x)\, y = q(x)$$

2. الحصول على معامل التكامل بواسطة حل تكامل P(x ورفعه كأس الصيغة

$$e^{\int P(x)\, dx} = I.F$$

3. نضرب كلا الطرفين للمعادلة الأولى بـ I.F لتصبح

$$e^{\int P(x)\, dx} \frac{dy}{dx} + y\, P(x)\, e^{\int P(x)\, dx} = q(x)\, e^{\int P(x)\, dx}$$

4. نتحقق من ان الجزء الايسر من المعادلة يصبح (product rule) وكتابته بهذه الصيغة

$$\boxed{\left(I.F \cdot y(x)\right)'}$$

$$\boxed{\frac{d}{dx}\left(I.F \cdot y(x)\right)}$$

5. تكامل الطرفين للحصول على الحل النهائي

$$\boxed{I.F \cdot y(x) = \int q(x) \cdot I.F\, dx + C}$$

EX1/ Solve the Linear D.E. $\frac{dy}{dx} = \left(\frac{1}{1+x^3}\right) - \left(\frac{3x^2}{1+x^3}\right) y$

① Rewrite
$$\frac{dy}{dx} + \left(\frac{3x^2}{1+x^3}\right) y = \frac{1}{1+x^3}$$

② Compare
$$\frac{dy}{dx} + P(x)\, y = q(x)$$

$\therefore\ P(x) = \frac{3x^2}{1+x^3}$, $q(x) = \frac{1}{1+x^3}$

③ Find the integrating factor (I.F.) $= e^{\int P(x)\, dx}$

$\therefore\ \text{I.F.} = e^{\int \frac{3x^2}{1+x^3}\, dx} = e^{\ln(1+x^3)}$

$\therefore\ \boxed{\text{I.F.} = 1+x^3}$

④ Rewrite
$$\frac{d\,(y \cdot \text{I.F.})}{dx} = q(x) \cdot \text{I.F.}$$

$$\therefore\ \frac{d\,(y * (1+x^3))}{dx} = \frac{1}{\cancel{(1+x^3)}} * \cancel{\text{I.F.}}\ \cancel{(1+x^3)}$$

⑤ Integrating both sides

$$y\,(1+x^3) = x$$

$$\therefore\ \boxed{y = \frac{x}{1+x^3}}$$

Ex2/ Solve the differential equation

$$\frac{dy}{dx} + 2y = 4e^{-2x}$$

Sol

$P(x) = 2$

$g(x) = 4e^{-2x}$

$$I.F. = e^{\int P(x)\,dx} = e^{\int 2\,dx}$$

$$\boxed{I.F. = e^{2x}}$$

نضرب كلا الطرفين بعامل التكامل

$$\frac{dy}{dx}(e^{2x}) + 2y(e^{2x}) = 4e^{-2x}(e^{2x})$$

$$\boxed{\frac{dy}{dx}e^{2x} + 2e^{2x}y} = 4$$

Remember the product rule

$$\frac{d}{dx}(f(x)\,g(x)) = f'(x)\,g(x) + f(x)\,g'(x)$$

Assume

$$f'(x) = \frac{dy}{dx}, \quad g(x) = e^{2x}$$

$$f(x) = y, \quad g'(x) = 2e^{2x}$$

substitute them in the product rule

$$\frac{d}{dx}(ye^{2x}) = \boxed{\frac{dy}{dx}e^{2x} + 2e^{2x}y}$$

$$\therefore \quad \frac{d}{dx}(ye^{2x}) = 4$$

Integrating $\int \frac{d}{dx}(ye^{2x})\,dx = \int 4\,dx$

$$ye^{2x} = 4x + C$$

$$\boxed{y = \frac{4x + C}{e^{2x}}}$$ General Solution

Ex3/ Solve the D.E. $\frac{dy}{dx} + 3x^2 y = 6x^2$

4

Sol/ It is linear since it has the form

$$\frac{dy}{dx} + p(x)\, y = q(x)$$

$\therefore$ $\boxed{p(x) = 3x^2}$, $\boxed{q(x) = 6x^2}$

$$I.F. = e^{\int 3x^2\, dx} =$$

$$\boxed{I.F. = e^{x^3}}$$

Multiply both sides by I.F.

$$e^{x^3} \frac{dy}{dx} + 3x^2 e^{x^3} y = 6x^2 e^{x^3}$$

$$\frac{d}{dx}\left[e^{x^3} \cdot y \right] = 6x^2 e^{x^3}$$

$$\int e^{x^3}\, dy = \int 6x^2 e^{x^3}\, dx$$

$$e^{x^3} \cdot y = 2e^{x^3} + C$$

$$\boxed{y = \frac{2e^{x^3} + C}{e^{x^3}}}$$

General solution

EX4/ Find the Solution of the initial-value problem

$$x^2 \frac{dy}{dx} + xy = 1 \quad , \; x > 0 \; , \; y(1) = 2$$

5

Sol/ $x^2 y' + xy = 1 \;\Big] * \frac{1}{x^2}$

$$y' + \frac{1}{x} y = \frac{1}{x^2}$$

$\therefore \boxed{P(x) = \frac{1}{x}} \;,\; \boxed{q(x) = \frac{1}{x^2}}$

$$I.F. = e^{\int \frac{1}{x} dx} = e^{\ln(x)}$$

$$\boxed{I.F. = x}$$

$$y' + \frac{1}{x} y = \frac{1}{x^2} \Big] * x$$

$$x y' + y = \frac{1}{x}$$

or $(xy)' = \frac{1}{x}$

$$xy = \int \frac{1}{x} dx = \ln x + C$$

$$\boxed{y = \frac{\ln x + C}{x}}$$

Since $y(1) = 2 \Rightarrow 2 = \frac{\ln(1) + C}{1} = C \;,\; \boxed{\therefore C = 2}$

$\therefore \boxed{y = \frac{\ln(x) + 2}{x}}$ Solution/