



Bernoulli Equation

1. Theory

A Bernoulli differential equation can be written in the following standard form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where $n \neq 1$ (the equation is thus nonlinear).

To find the solution, change the dependent variable from y to z , where $z = y^{1-n}$. This gives a differential equation in x and z that is linear, and can be solved using the integrating factor method.

Note: Dividing the above standard form by y^n gives:

$$\frac{1}{y^n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

i.e. $\frac{1}{(1-n)} \frac{dz}{dx} + P(x)z = Q(x)$

(where we have used $\frac{dz}{dx} = (1-n)y^{-n} \frac{dy}{dx}$).

EXERCISE 1.

The general form of a Bernoulli equation is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,$$

where P and Q are functions of x , and n is a constant. Show that the transformation to a new dependent variable $z = y^{1-n}$ reduces the equation to one that is linear in z (and hence solvable using the integrating factor method).

Bernoulli Differential Equation: lec. 4

$$\boxed{\bar{y} + P(x) \cdot y = Q(x) \cdot y^n} \text{ general form}$$

* طريقة اكل: هي ان تقوم بعدد من الخطوات لتحويلها الى معادلة تفاضلية خطية (linear eq.)

* خطوات اكل:

1- تقسم طرفي المعادلة على (y^n)

$$\textcircled{1} \frac{\bar{y}}{y^n} + P(x) \cdot \frac{1}{y^{n-1}} = Q(x) \text{ ----- } \textcircled{1}$$

2- نعرفه ان $(Z = \frac{1}{y^{n-1}})$

$$\textcircled{2} \text{ let } Z = \frac{1}{y^{n-1}} \Rightarrow Z = y^{1-n} \text{ ----- } \textcircled{2}$$

طابقاً

بالتعويض $Z = y^{1-n}$

3- مشتقة الـ (Z) بالنسبة لـ (y)

$$\textcircled{3} \bar{Z} = (1-n) \cdot y^{-n} \cdot \bar{y} \text{ ----- } \textcircled{3}$$

نقوم بـ (2) و (3) في (1)

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$$\frac{1}{n-1} \cdot \bar{Z} + P(x) \cdot Z = Q(x) \quad (\text{linear equation})$$

$$\text{Ex 1: solve } [\bar{y} + 2xy = -xy^4]$$

Sol:

$$[\bar{y} + 2xy = -xy^4] \div y^4$$

$$\frac{\bar{y}}{y^4} + \frac{2x}{y^3} = -x \quad \text{--- (1)}$$

$$\text{let } \frac{1}{y^3} = Z \implies Z = y^{-3} \quad \text{--- (2)}$$

$$\bar{Z} = -3y^{-4} \cdot \frac{dy}{dx}$$

$$\bar{Z} = -3y^{-4} \cdot \bar{y} \longrightarrow \bar{Z} = -3 \frac{\bar{y}}{y^4} \quad \text{--- (3)}$$

نضرب المعادلة (-3) * (1)

$$-3 \frac{\bar{y}}{y^4} - \frac{6x}{y^3} = 3x \quad \text{--- (4)}$$

sub 2 & 3 in (4)

$$\bar{Z} - 6xZ = 3x \quad (\text{linear D.E.})$$

(2)

$$P(x) = -6x, \quad Q(x) = 3x$$

$$IF = \frac{\int P(x) dx}{e^{\int Q(x) dx}} = \frac{\int -6x dx}{e^{-3x^2}} = \frac{-3x^2}{e^{-3x^2}}$$

$$y \cdot IF = \int [Q(x) + IF dx] + C$$

$$y \cdot e^{-3x^2} = \int (3x \cdot e^{-3x^2}) dx + C$$

$$y \cdot e^{-3x^2} = -\frac{1}{2} \int (-2 \cdot 3x \cdot e^{-3x^2}) dx + C$$

$$y \cdot e^{-3x^2} = -\frac{1}{2} e^{-3x^2} + C$$

Ex2: Solve $\bar{y} + 2xy = xy^3$

Sol: $[\bar{y} + 2xy = xy^3] \div y^3 \rightarrow n=3$

$$\frac{\bar{y}}{y^3} + \frac{2x}{y^2} = x \quad \text{--- (1)}$$

let $Z = \frac{1}{y^2}$ or $Z = y^{1-n} \rightarrow Z = y^{-2}$ --- (2)

$$\bar{Z} = -2y^{-3} \cdot \bar{y} \rightarrow \bar{Z} = \frac{-2\bar{y}}{y^3} \quad \text{--- (3)}$$

(4) نضرب معادلة (1) ونعبرها بمعادلة (2)

$$-2 \frac{\bar{y}}{y^3} - \frac{4x}{y^2} = -2x \quad \text{--- (4)}$$

نوعان (2) و (3) من (4)

$$\bar{z} - 4x z = -2x \quad (\text{linear D.E.})$$

$$P(x) = -4x$$

$$Q(x) = -2x$$

$$\text{IF} = e^{\int -4x dx} = e^{-2x^2}$$

$$\therefore y \text{IF} = \int (Q(x) \cdot \text{IF} dx) + C$$

$$y \cdot e^{-2x^2} = \int (-2x \cdot e^{-2x^2}) + C$$

$$y e^{-2x^2} = \frac{1}{2} \int (2 \cdot -2x \cdot e^{-2x^2}) + C$$

$$y e^{-2x^2} = \frac{1}{2} \cdot e^{-2x^2} + C$$

(4)