**2) Homogeneous Differential Equation**

## 1st Order DE - Homogeneous Equations

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous and of the same degree

Solution :

1. Use the transformation to : $\boxed{y = vx \;\Rightarrow\; dy = v\,dx + x\,dv}$
2. The equation become separable equation:
$$\boxed{P(x,v)dx + Q(x,v)dv = 0}$$
3. Use solution method for separable equation
$$\boxed{\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(v)}{g_1(v)}dv = C}$$
4. After integrating, $v$ is replaced by $y/x$

3) Exact Differential Equations

## 1st Order DE – Exact Equation

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if : $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

The solutions are given by the implicit equation $\boxed{F(x,y)=C}$
where : $\partial F/\partial x = M(x,y)$ and $\partial F/\partial y = N(x,y)$

Solution :

1. Integrate either $M(x,y)$ with respect to $x$ or $N(x,y)$ to $y$.
   Assume integrating $M(x,y)$, then :

$$F(x,y) = \int M(x,y)dx + \theta(y)$$

2. Now : $\dfrac{\partial F}{\partial y} = \dfrac{\partial}{\partial y}\left[\int M(x,y)dx\right] + \theta'(y) = N(x,y)$

   or : $\theta'(y) = N(x,y) - \dfrac{\partial}{\partial y}\left[\int M(x,y)dx\right]$

lec. 2

2- Homogeneous D.E. المعادلات التفاضلية المتجانسة

The D.E. is homogeneous if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

EX1: Solve $\frac{dy}{dx} = \frac{x-y}{x+y}$ --- (dividing by x).

Sol:

$$\frac{dy}{dx} = \frac{\frac{x-y}{x}}{\frac{x+y}{x}}$$

$$\frac{dy}{dx} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \longrightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

إذن هي من نوع المتجانسة

hence: let $\boxed{v = \frac{y}{x} \longrightarrow y = vx} \rightarrow$ فرضية

مشتقة الفرضية $\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$

$$\therefore \frac{dy}{dx} = \frac{1-v}{1+v}$$

$$v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$

$$\frac{x\,dv}{dx} = \frac{1-v}{1+v} - v$$

$$\frac{x\,dv}{dx} = \frac{1-2v-v^2}{1+v}$$

$$\int \frac{dx}{x} = \int \frac{1+v}{1-2v-v^2}\,dv$$

$$\ln|x| = -\frac{1}{2}\int \frac{-2(1+v)}{1-2v-v^2}\,dv$$

$$\ln x = -\frac{1}{2}\ln(1-2v-v^2) + c$$

$$\ln|x| = -\frac{1}{2}\ln\left|\left(1-2\frac{y}{x}-\left(\frac{y}{x}\right)^2\right)\right| + c$$

نعوض بدل كل $v = \frac{y}{x}$

Ex2 : Solve $(x^2+3y^2)\,dx - 2xy\,dy = 0$

Sol: $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy} \Big] \div x^2$

$$\frac{dy}{dx} = \frac{1+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right) \text{ ------ } 1$$

∴ homogeneous

let $\boxed{v = \frac{y}{x} \longrightarrow y = vx}$ & $\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx} \cdots 2}$

الفرضية        مشتقة الفرضية

sub eq 2 in eq. 1

$$v + x\frac{dv}{dx} = \frac{1+3v^2}{2v}$$

$$\frac{x\,dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$\frac{x\,dv}{dx} = \frac{1+3v^2-2v^2}{2v}$$

$$\frac{dx}{x} = \frac{2v}{1+3v^2-2v^2}dv \longrightarrow \int\frac{dx}{x} = \int\frac{2v\,dv}{1+v^2}$$

$$\ln|x| = \ln|(1+v^2)| + c$$

$$\ln\left|\frac{x}{1+v^2}\right| = c \quad , \text{ sub } v = \frac{y}{x}$$

$$\ln\left|\frac{x}{1+\frac{y}{x}}\right| = c$$

Ex3: Solve $x\left[\frac{dy}{dx} - \tan\frac{y}{x}\right] = y$

dividing By (x)

Sol:

$$\frac{dy}{dx} - \tan\frac{y}{x} = \frac{y}{x}$$

$$\frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x} \quad ---- \ 1$$

$$v = \frac{y}{x} \longrightarrow \boxed{y = vx \quad \& \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad --- \ 2}$$

sub. 2 in 1

$$v + x\frac{dv}{dx} = \tan v + v$$

$$\frac{x\,dv}{dx} = \tan v + \cancel{v} - \cancel{v}$$

$$\int\frac{dx}{x} = \int\frac{dv}{\tan v}$$

$$\int\frac{dx}{x} = \int\frac{dv}{\frac{\sin v}{\cos v}}$$

$$\int\frac{dx}{x} = \int\frac{\cos v}{\sin v}\,dv$$

$$\ln|x| = \ln|\sin v| + c$$

$$\ln|x| = \ln\left|\sin\frac{y}{x}\right| + c$$

$$\ln\left|\sin\frac{y}{x}\right| - \ln|x| + c \quad 3$$

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## 3- Exact Differential Equations

The D.E. is exact if $M\,dx + N\,dy = 0$ & $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

EX1: Solve $(\underbrace{2xy}_{M}\,dx + (\underbrace{x^2 + \cos y}_{N})\,dy = 0)$ ?

Sol:-

$$\frac{\partial M}{\partial y} = 2x \quad , \quad \frac{\partial N}{\partial x} = 2x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$$

$\therefore$ D.E. is exact.

الشكل العام للحل هو : $\boxed{f(x,y) = \int M\,dx + g(y)}$

نعوض في

$$\int M\,dx = \int 2xy\,dx = x^2 y$$

$$\therefore f = x^2 y + g(y) \quad ---- \text{①}$$

لاستخراج قيمة ال $g(y)$ نشتق المعادلة ① بالنسبة لل $(y)$ ونساويها لل $(N)$ لان $(N)$ هي دالة لل $(y)$

$$\therefore \frac{\partial f}{\partial y} = x^2 + g'(y) \quad ---- \text{②}$$

$$\frac{\partial f}{\partial y} = N = x^2 + \cos y \quad --- \text{③}$$

نعوض ② في ③

$$\cancel{x^2} + \cos y = \cancel{x^2} + g'(y)$$

$$\therefore g'(y) = \cos y$$

بالتكامل $g(y) = \sin y + c$

$$\therefore f = x^2 y + \sin y + c$$

EX2: Solve $\dfrac{dy}{dx} = \dfrac{xy^2-1}{1-x^2y}$

Sol/ $M\,dx + N\,dy = 0$ ;

$(xy^2-1)\,dx + (-(1-x^2y))\,dy = 0$

$(xy^2-1)\,dx + (x^2y-1)\,dy = 0$

$M = xy^2-1$

$N = x^2y-1$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2xy \\ \frac{\partial N}{\partial x} &= 2xy \end{aligned}\right] \therefore \text{ exact type}$$

$\boxed{f(x,y) = \int M + g(y)} \longrightarrow$ الحل العام

$\int M = \int xy^2 - 1\,dx = \dfrac{x^2y^2}{2} - x \longrightarrow$ تعوض في الحل العام

$\therefore f = \dfrac{x^2y^2}{2} - x + g(y)$

$\dfrac{\partial f}{\partial y} = x^2y + g'(y)$

$\dfrac{\partial f}{\partial y} = N$

$\therefore x^2\cancel{y} + g'(y) = x^2\cancel{y} - 1$

$g'(y) = -1$

$\int g'(y) = g(y) = -y + C$

$\therefore f = \dfrac{x^2y^2}{2} - x - y + C$