2) Homogeneous Differential Equation

1st Order DE - Homogeneous Equations

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous and of the same degree

Solution :

1. Use the transformation to : $y = vx \Rightarrow dy = v\,dx + x\,dv$
2. The equation become separable equation:
   $$P(x,v)dx + Q(x,v)dv = 0$$
3. Use solution method for separable equation
   $$\int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(v)}{g_1(v)}dv = C$$
4. After integrating, v is replaced by y/x

مدرس المادة

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3) Exact Differential Equations

1st Order DE – Exact Equation

The differential equation $M(x,y)dx + N(x,y)dy = 0$ is an exact equation if : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The solutions are given by the implicit equation $\boxed{F(x,y) = C}$
where : $\partial F/\partial x = M(x,y)$ and $\partial F/\partial y = N(x,y)$

Solution :

1. Integrate either $M(x,y)$ with respect to x or $N(x,y)$ to y.
   Assume integrating $M(x,y)$, then :

$$F(x,y) = \int M(x,y)dx + \theta(y)$$

2. Now : $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)dx\right] + \theta'(y) = N(x,y)$

   or : $\theta'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M(x,y)dx\right]$

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lec. 2

2- Homogeneous D.E. المعادلات التفاضلية المتجانسة

The D.E. is homogeneous if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

EX1: Solve $\frac{dy}{dx} = \frac{x-y}{x+y}$ --- (dividing by X).

Sol:

$$\frac{dy}{dx} = \frac{\frac{x-y}{x}}{\frac{x+y}{x}}$$

$$\frac{dy}{dx} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \longrightarrow \frac{dy}{dx} = f\left(\frac{y}{x}\right)$$ إذن هي من نوع المتجانسة

hence: let $V = \frac{y}{x} \longrightarrow y = VX$ → فرضية -

مشتقة الفرضية $\frac{dy}{dx} = V + X\frac{dV}{dX}$

$$\therefore \frac{dy}{dX} = \frac{1-V}{1+V}$$

$$V + X\frac{dV}{dX} = \frac{1-V}{1+V}$$

$$\frac{X\,dV}{dX} = \frac{1-V}{1+V} - V$$

$$\frac{X\,dV}{dX} = \frac{1-2V-V^2}{1+V}$$

$$\int \frac{dX}{X} = \int \frac{1+V}{1-2V-V^2}\,dV$$

$$\ln|X| = -\frac{1}{2}\int \frac{-2(1+V)}{1-2V-V^2}\,dV$$

$$\ln X = -\frac{1}{2}\ln(1-2V-V^2) + C$$

$$\ln|X| = \frac{-1}{2}\ln\left|\left(1-2\frac{y}{x}-\left(\frac{y}{x}\right)^2\right)\right| + C$$

نعوض بدل كل $V = \frac{y}{x}$

Ex2 : Solve $(x^2+3y^2)\,dx - 2xy\,dy = 0$

Sol: $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$] $\div x^2$

$$\frac{dy}{dx} = \frac{1+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right) \quad \text{------ (1)}$$

∴ homogeneous

let $v = \frac{y}{x} \rightarrow y = vx$ & $\frac{dy}{dx} = v + x\frac{dv}{dx}$ .... (2)

الفرضية — مشتقة الفرضية

sub eq (2) in eq. (1)

$$v + x\frac{dv}{dx} = \frac{1+3v^2}{2v}$$

$$\frac{x\,dv}{dx} = \frac{1+3v^2}{2v} - v$$

$$\frac{x\,dv}{dx} = \frac{1+3v^2-2v^2}{2v}$$

$$\frac{dx}{x} = \frac{2v}{1+3v^2-2v^2}\,dv \longrightarrow \int\frac{dx}{x} = \int\frac{2v\,dv}{1+v^2}$$

$$\ln|x| = \ln|(1+v^2)| + c$$

$$\ln\left|\frac{x}{1+v^2}\right| = c \quad , \text{ sub } v = \frac{y}{x}$$

$$\ln\left|\frac{x}{1+\frac{y}{x}}\right| = c$$

EX3: Solve $x\left[\frac{dy}{dx} - \tan\frac{y}{x}\right] = y$

dividing By (x)

Sol:

$$\frac{dy}{dx} - \tan\frac{y}{x} = \frac{y}{x}$$

$$\frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x} \quad \text{----} \ (1)$$

$$v = \frac{y}{x} \longrightarrow \boxed{y = vx \quad \& \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{---} \ (2)}$$

sub. (2) in (1)

$$v + x\frac{dv}{dx} = \tan v + v$$

$$\frac{x\,dv}{dx} = \tan v + \cancel{v} - \cancel{v}$$

$$\int \frac{dx}{x} = \int \frac{dv}{\tan v}$$

$$\int \frac{dx}{x} = \int \frac{dv}{\frac{\sin v}{\cos v}}$$

$$\int \frac{dx}{x} = \int \frac{\cos v}{\sin v}\,dv$$

$$\ln|x| = \ln|\sin v| + c$$

$$\ln|x| = \ln\left|\sin\frac{y}{x}\right| + c$$

$$\ln\left|\sin\frac{y}{x}\right| - \ln|x| + c \quad (3)$$

③ Exact Differential Equations

The D.E. is exact if $M\,dx + N\,dy = 0$ & $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

EX1: Solve $(\underbrace{2xy}_{M}\,dx + (\underbrace{x^2 + \cos y}_{N})\,dy = 0)$?

Sol:-

$$\frac{\partial M}{\partial y} = 2x \quad , \quad \frac{\partial N}{\partial x} = 2x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x$$

$\therefore$ D.E. is exact.

الشكل العام للحل هو: $f(x,y) = \int M\,dx + g(y)$

نعوض في

$$\int M\,dx = \int 2xy\,dx = x^2 y$$

$$\therefore f = x^2 y + g(y) \quad ---- \quad ①$$

لاستخراج قيمة ال $g(y)$ نشتق المعادلة ① بالنسبة لل (y) ونساويها لل (N) لان (N) هي دالة لل (y)

$$\therefore \frac{\partial f}{\partial y} = x^2 + g'(y) \quad ---- \quad ②$$

$$\frac{\partial f}{\partial y} = N = x^2 + \cos y \quad --- \quad ③$$

نعوض ③ في ②

$$\cancel{x^2} + \cos y = \cancel{x^2} + g'(y)$$

$$\therefore g'(y) = \cos y$$

بالتكامل $g(y) = \sin y + c$

$$\therefore f = x^2 y + \sin y + c$$

Ex2: Solve $\frac{dy}{dx} = \frac{xy^2-1}{1-x^2y}$

Sol/ $M\,dx + N\,dy = 0$;

$(xy^2-1)\,dx + (-(1-x^2y))\,dy = 0$

$(xy^2-1)\,dx + (x^2y-1)\,dy = 0$

$M = xy^2-1$

$N = x^2y-1$

$\frac{\partial M}{\partial y} = 2xy$

$\frac{\partial N}{\partial x} = 2xy$

∴ exact type

$f(x,y) = \int M + g(y)$ → الحل العام

$\int M = \int xy^2-1\,dx = \frac{x^2y^2}{2} - x$ → تعوض في الحل العام

∴ $f = \frac{x^2y^2}{2} - x + g(y)$

$\frac{\partial f}{\partial y} = x^2y + g'(y)$

$\frac{\partial f}{\partial y} = N$

∴ $x^2y + g'(y) = x^2y - 1$

$g'(y) = -1$

$\int g'(y) = g(y) = -y + C$

∴ $f = \frac{x^2y^2}{2} - x - y + C$